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The Improved (G'/G)-Expansion Method to the Generalized Burgers-Fisher Equation

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Abstract: In this article, the improved (G'/G)-expansion method has been implemented to generate travelling wave solutions, where $G(\eta)$ satisfies the second order nonlinear ordinary differential equation. To show the advantages of the method, the Generalized Burgers-Fisher equation has been investigated. Nonlinear partial differential equations have many potential applications in mathematical physics and engineering sciences. Some of our solutions are in good agreement with already published results for a special case and others are new. The solutions in this work may express a variety of new features of waves. Furthermore, these solutions can be valuable in the theoretical and numerical studies of the considered equation.

Keywords: The Improved (G'/G)-Expansion Method, The Generalized Burger's-Fisher Equation, Traveling Wave Solutions, Nonlinear Evolution Equations

1. Introduction

Nonlinear partial differential equations (NLPDEs) are widely used to describe complex physical phenomena in different fields of study in mathematical physics, engineering science and other areas of natural science [1]. In Particular, the generalized Burger's-Fisher equation has a wide range of applications in mathematical-physics, engineering science, physics, chemistry etc.

Exact travelling wave solutions of NLPDEs play an important role in the study of nonlinear physical phenomena. Looking for exact solutions to nonlinear evolution equations (NLEEs) has long been a major concern for both mathematicians and physicists. These solutions may well describe various phenomena in physics and other fields [1]. But unlike LPDE, NPDEs are difficult to study because there are almost no general techniques that work for all NPDEs, and usually each individual equation has to be studied as a separate problem. Therefore, many authors have been introducing different techniques to obtain exact traveling wave solutions for non-linear evolution equations (NLEEs) for the past many years. Recently, several direct methods such as Exp-function method [2, 3], sine-cosine method [4], tanh-coth [7], tanh method [6], auxiliary equation method [7], the first integral method [8], Improved Adomian decomposition method [9], Variational iteration method [10], the Adomian Decomposition Method [11] and others have been proposed to obtain exact solutions of nonlinear partial differential equations. Using these methods many exact solutions, including the solitary wave solutions, shock wave solutions and periodic wave solutions are obtained for some kinds of nonlinear evolution equations.

Another important method presented to construct exact solutions of nonlinear PDEs is the basic (G'/G)-expansion method. The concept of this method was first proposed by Wang et al. [12], consequently, many researchers applied the (G'/G)-expansion method to solve different kinds of NLEEs [13, 14, 15, 16]. More recently, Zhang et al. [17] extended the basic (G'/G)-expansion method which is called the improved (G'/G)-expansion method to establish abundant traveling wave solutions of nonlinear PDEs. Many researchers applied the improved (G'/G)-expansion method to different nonlinear PDEs [18, 19, 20, 21, 23]. It has been shown that this method is straightforward, concise, basic and effective.

The importance of our current work is, in order to generate abundant traveling wave solutions, the generalized Burgers-Fisher equation has been considered by applying the improved (G'/G)-expansion method. The structure of this paper is organized as follows. In section 2, the improved (G'/G)-expansion method is discussed. Application of improved (G'/G)-expansion method to the generalized Burgers-Fisher equation is presented in Section 3. In Section 4 ends this work with a brief conclusion.

2. The Improved (G'/G)-Expansion Method

We consider that the given Nonlinear Partial Differential Equation in the form of

$$P\left(u,u_x,u_t,u_{xt},u_{xx},\ldots\right)=0 \tag{1}$$

where P is a polynomial in its arguments, which includes nonlinear terms and the highest order derivatives, the subscript stands for partial derivatives and $u(x,t)$ is the unknown function.

REMARK1: Nonlinear evolution equation (NLEE) is a NPDE which is dependent of a time t.

EXAMPLE

1. $u_t = uu_x + u_{xx}$ (Burger's equation)
2. $u_t = uu_x + u_{xx} + u(1-u)$ (Burger-Fisher equation)
3. $u_t + pu^n u_x + qu_{xx} + ru(1-u^n)=0$, (Generalized Burger-Fisher equation), and so on

Travelling Wave Solution

A travelling wave solution of a NPDE in two variables (x,t) is a solution of the form $u(x,t)=U(\eta)$, $(\eta = x - ct)$, c is a speed of traveling wave) where $U(\eta)$ is an arbitrary differentiable function of η.

The traveling wave transformations

Combining the real variables x and t by a wave variable $\eta = kx+\omega t$

$$u(x,t)=U(\eta) \tag{2}$$

where ω is the speed of the traveling wave.

The traveling wave transformations (2) converts (1) into an ordinary differential equation (ODE)

$$Q(U,kU',\omega U',k\omega U'',k^2U'',\ldots)=0 \tag{3}$$

Where Q is a polynomial in U and its derivatives; the superscripts indicate the ordinary derivatives with respect to η.

Traveling wave solutions

The solution of (3) can be expressed as follows:

$$U(\eta)=\sum_{i=0}^{m}\alpha_i(\mathrm{G'/G})^i+\sum_{i=1}^{m}\beta_i(\mathrm{G'/G})^{-i} \tag{4}$$

Where $\alpha_i (i=0,1,2,...,m)$, $\beta_i (i=1,2,...,m)$ are arbitrary constants to be determined and either α_m *or* β_m can be zero but both can't be zero at the same time [21] and $G=G(\eta)$ satisfies the following second order nonlinear ordinary differential equation with constant coefficients:

$$GG''=\lambda GG'+\mu G^2+V(G')^2 \tag{5}$$

Where the prime stands for derivative with respect to η and $\lambda,\mu,$ and V are real parameters.

The Cole-Hopf transformation $\phi(\eta)=\ln(G(\eta))_\eta$ transforms (5) into the generalized Riccati type equation in terms of $\phi(\eta)$:

$$\phi'(\eta)=\mu+\lambda\phi(\eta)+(v-1)\phi^2(\eta) \tag{6}$$

where $\phi(\eta)=\left(G'(\eta)/G(\eta)\right)$. The generalized Riccati equation has 25 distinct solutions [24] and (see Appendix I for details)

Note that from (4), (5) and (6) it follows:

$$U'(\eta)=\left(\mu+\lambda(\mathrm{G'/G})+(V-1)(\mathrm{G'/G})^2\right)\left(\sum_{i=0}^{m}i\alpha_i(\mathrm{G'/G})^{i-1}-\sum_{i=1}^{m}i\beta_i(\mathrm{G'/G})^{-i-1}\right)$$

$$U''(\eta)=\left(\mu+\lambda(\mathrm{G'/G})+(V-1)(\mathrm{G'/G})^2\right)\left(\lambda+2(V-1)(\mathrm{G'/G})\right)\left(\sum_{i=0}^{m}i\alpha_i(\mathrm{G'/G})^{i-1}-\sum_{i=1}^{m}i\beta_i(\mathrm{G'/G})^{-i-1}\right)-$$

$$\left(\mu+\lambda(\mathrm{G'/G})+(V-1)(\mathrm{G'/G})^2\right)\left(\sum_{i=0}^{m}i(i-1)\alpha_i(\mathrm{G'/G})^{i-2}\right) \tag{7}$$

And so forth, where the prime denotes derivative with respect to η.

Now, to determine $u(x,t)$ explicitly we follow the following steps:

Step 1: transforming (1) into (3) (ODE) using traveling wave transformations in (2).

Step 2: substitute (4) including (6) and (7) into (3) to determine the positive integer m, taking the homogeneous balance between the highest order nonlinear term and the derivative of the highest order appearing in (3).

Step 3: Substitute (4) including (6) and (7) into (3) with the value of m obtained in Step 2, to obtain polynomials in $(G'/G)^i\ (i=0,1,\ldots,m)$ and $(G'/G)^{-i}\ (i=1,2,\ldots,m)$ subsequently, we collect each coefficient of the resulted polynomials to zero, yields a set of algebraic equations for $\alpha_0,\alpha_1,\ldots\alpha_m,\beta_1,\ldots\beta_m,\ \omega$ *and* μ.

Step 4: Suppose that the value of the constants $\alpha_i(i=01,\ldots m),\beta_i(i=1,2,\ldots m),\mu$ and ω can be found by solving the algebraic equations obtained in Step 5. Since the general solutions of (6) are known to us, inserting the values of $\alpha_i(i=01,2,\ldots m),\beta_i(i=1,2,\ldots m),\mu$ and ω into (4), we obtain more general type and new exact traveling wave solutions of the nonlinear partial differential equation (1).

3. Application of the Method

In this section, we will apply the improved $\left(G'/G\right)$ expansion method to construct many new and more general traveling wave solutions of the generalized Burger-fisher equation in [22].

Now considering the generalized Burgers-Fisher equation with higher-order nonlinear terms

$$u_t + pu^n u_x + qu_{xx} + ru\left(1-u^n\right) = 0 \tag{8}$$

where $n>1$, and $p,\ r,$ and q are real parameters.

To look for travelling wave solutions of (13), we use the wave transformation (2) and change (3) into the form of an ODE

$$\omega U' + pkU^nU' + qk^2U'' + rU(1-U^n) = 0 \tag{9}$$

To determine the positive integer m in (4), we need special transformation as mentioned in [22]

$$U(\eta) = V^{\frac{1}{n}}(\eta),\quad \eta = kx + \omega t\ \ (u(x,t) = U(\eta)) \tag{10}$$

Inserting (15) and its first and second derivative reduce (14) into

$$\omega nVV' + pnkV^2V' + qk^2(1-n)(V')^2 + qk^2nVV'' + rn^2V^2 - rn^2V^3 = 0 \tag{11}$$

Now the homogeneous balance between VV'' and V^2V' in (16) can be found from

$$vv'' = v^2v' \quad \Rightarrow 2m+2 = 3m+1 \quad \Rightarrow m = 1$$

Therefore, the solution of (16) becomes

$$V(\eta) = \alpha_0 + \alpha_1\left(\frac{G'}{G}\right) + \beta_1\left(\frac{G'}{G}\right)^{-1} \quad \alpha_1 \neq 0 \text{ or } \beta_1 \neq 0 \tag{12}$$

Now inserting (12) and its first and second derivative with (5) and (6) into (11) we obtained polynomials in $(G'/G)^i\ \left(i=0,1,2,\ldots m\right)$ and $(G'/G)^{-i}\ \left(i=1,2,3,\ldots m\right)$. Subsequently, we collect each coefficient of the resulted polynomials to zero, yields a set of algebraic equations for $\alpha_0,\alpha_1,\beta_1,\mu$ and ω as follows:

$$\begin{aligned}\left(G'/G\right)^4 :& -2qk^2na_1^{\ 2}v + pnka_1^{\ 3}v + qk^2na_1^{\ 2} + qk^2na_1^{\ 2}v^2 + qk^2a_1^{\ 2} + qk^2a_1^{\ 2}v^2 - pnka_1^{\ 3} - \\ & 2qk^2a_1^{\ 2}v = 0\end{aligned}$$

$$\begin{aligned}\left(G'/G\right)^3 :& -rn^2a_1^{\ 3} + \omega na_1^{\ 2}v + pnka_1^{\ 3}\lambda + 2qk^2a_1^{\ 2}v\lambda + 2pnka_0a_1^{\ 2}v + qk^2na_1^{\ 2}v\lambda + 2qk^2na_0a_1 - \\ & 2qk^2a_1^{\ 2}\lambda - \omega na_1^{\ 2} - qk^2na_1^{\ 2}\lambda - 2pnka_0a_1^{\ 2} + 2qk^2na_0a_1v^2 - 4qk^2na_0a_1v = 0\end{aligned}$$

$$\begin{aligned}\left(G'/G\right)^2 :& o\omega na_1^{\ 2}\lambda + 4qk^2a_1\beta_1v - 2qk^2a_1^{\ 2}\mu + 4qk^2na_1v^2\beta_1^{\ 2} + 3qk^2na_0a_1\lambda + 2qk^2a_1^{\ 2}v\mu^2 - pnka_1^{\ 2} + \\ & pnka_1^{\ 3}\mu - 8qk^2na_1\beta_1v - 2qk^2a_1\beta_1 - 3rn^2a_0a_1^{\ 2} - pnka_0^{\ 2}a_1 + 2pnka_0a_1^{\ 2}\lambda - na_0a_1 - 3qk^2na_0a_1\lambda + \\ & pnka_0^{\ 2}a_1v + qk^2a_1^{\ 2}\lambda^2 + \omega na_0a_1v - 2qk^2a_1v^2\beta_1^{\ 2} + rn^2a_1^{\ 2} + pnka_1^{\ 2}\beta_1v + 4qk^2na_1\beta_1 = 0\end{aligned}$$

$$\begin{aligned}\left(G'/G\right)^1 :& 2qk^2na_0a_1v\mu + 4qk^2a_1\beta_1\lambda - 4qk^2a_1v\beta_1\lambda + 8qk^2na_1v\beta_1\lambda + \omega na_0a_1\lambda + qk^2na_0a_1\lambda^2 + pnka_0^{\ 2}a_1\lambda + \\ & 2rn^2a_0a_1 + \omega na_1^{\ 2}\mu - 3rn^2a_1^{\ 2}\beta_1 - qk^2na_1^{\ 2}\lambda\mu - 2qk^2na_0a_1\mu + 2pnka_0a_1^{\ 2}\mu - 8qk^2na_1\lambda\beta_1 - 3rn^2a_0^{\ 2}a_1 + \\ & pnka_1^{\ 2}\lambda\beta_1 + 2qk^2a_1^{\ 2}\lambda\mu = 0\end{aligned}$$

$$\left(G'/G\right)^0 : -4qk^2 a_1 v\beta_1\mu - qk^2 na_0\lambda\beta_1 + pnka_1{}^2\beta_1\mu + pnka_0{}^2 a_1\mu - pnka_0{}^2\beta_1 v - pnka_1\beta_1{}^2 v + 4qk^2 na_1\lambda^2\beta_1$$
$$-8qk^2 na_1\beta_1\mu + A^2 qk^2\beta_1^2 + rn^2 a_0{}^2 - rn^2 a_0{}^3 - 6A^2 rn^2 a_0 a_1\beta_1 + qk^2\beta_1^2 v^2 - 2qk^2\beta_1^2 v + qk^2 a_1^2\mu^2 + pnka_0{}^2\beta_1 +$$
$$pnka_1\beta_1^2 + \omega na_0\beta_1 - qk^2 n\beta_1^2 + 2rn^2 a_1\beta_1 + qk^2 na_0\beta_1 v\lambda + 8qk^2 na_1 v\beta_1\mu + qk^2 na_0 a_1\lambda\mu + \omega na_0 a_1\mu - \omega a_0\beta_1 v$$
$$-2qk^2 a_1\lambda^2\beta_1 + 4qk^2 a_1\mu\beta_1 - qk^2 n\beta_1^2 v^2 + 2qk^2 n\beta_1^2 v - qk^2 na_1{}^2\mu^2 = 0$$

$$\left(G'/G\right)^{-1} : qk^2 n\beta_1^2\lambda - 2qk^2\beta_1^2\lambda + 8qk^2 na_1\lambda\beta_1\mu - \omega n\beta_1^2 v + 2qk^2\beta_1^2\lambda v - 4qk^2 a_1\lambda\beta_1\mu + \omega n\beta_1^2$$
$$-pnka_1\beta_1^2\lambda + 2rn^2 a_0\beta_1 + 2qk^2 na_0\beta_1\mu v - 3rn^2 a_0{}^2\beta_1 - pnka_0{}^2\lambda\beta_1 - qk^2 n\beta_1^2 v\lambda - 2pnka_0\beta_1^2 v$$
$$-2qk^2 na_0\beta_1\mu - 3rn^2 a_1\beta_1^2 - \omega na_0\beta_1\lambda + 2pnka_0\beta_1^2 + qk^2 na_0\beta_1\lambda^2 = 0$$

$$\left(G'/G\right)^{-2} : -3rn^2 a_0\beta_1^2 + 4qk^2 na_1\mu^2\beta_1 + 2qk^2\beta_1^2 v\mu - pnk\beta_1^3 v + qk^2\beta_1^2\lambda^2 + rn^2\beta_1^2 - \omega na_0\beta_1\mu -$$
$$2qk^2\beta_1^2\mu - 2pnka_0\beta_1^2\lambda - pnka_0{}^2\mu\beta_1 - pnka_1\beta_1^2\mu + pnk\beta_1^3 - 2qk^2 a_1\mu^2\beta_1 - \omega n\beta_1^2\lambda +$$
$$3qk^2 na_0\mu\lambda\beta_1 = 0$$

$$\left(G'/G\right)^{-3} : rn^2\beta^3 + 2qk^2 na_0\beta_1\mu^2 - pnk\beta_1^3\lambda + 2qk^2\beta_1^2\lambda\mu + qk^2 n\beta_1^2\lambda\mu - \omega n\beta_1^2\mu - 2pnka_0\beta_1^2\mu = 0$$

$$\left(G'/G\right)^{-4} : \beta_1^2 k^2\mu^2 nq - \beta_1^3 k\mu np + \beta_1^2 k^2\mu^2 q = 0$$

Solving the systems of obtained algebraic equations with the aid of algebraic software Maple, we obtain three different sets of values.

Set 1:

$$\alpha_0 = \frac{k\lambda q(n+1)+np}{2np},\ \alpha_1 = 0,\ \beta_1 = \frac{1}{4}\frac{n^2p^2-(k\lambda q(n+1))^2}{pqkn(n+1)(1-v)},\ \mu = \frac{1}{4}\frac{n^2p^2-(k\lambda q(n+1))^2}{(n+1)^2(1-v)q^2k^2}$$

$$\omega = -\frac{(qr(n+1)^2+p^2)k}{(n+1)p} \text{ and } \lambda = \lambda \tag{13}$$

Set 2:

$$\alpha_0 = \frac{np-k\lambda q(n+1)}{2np}, \alpha_1 = \frac{kq(1-v)(n+1)}{np}, \beta_1 = 0, \mu = \frac{1}{4}\frac{n^2p^2-(k\lambda q(n+1))^2}{(n+1)^2(1-v)q^2k^2}$$

$$\omega = -\frac{(qr(n+1)^2+p^2)k}{(n+1)p} \text{ and } \lambda = \lambda \tag{14}$$

Set 3:

$$a_0 = 1/2, a_1 = kq(n+1)(1-v)/(np), \beta_1 = \frac{np}{16kq(n+1)(1-v)}, \lambda = 0, \mu = \frac{n^2p^2}{16(n+1)^2(1-v)k^2q^2},$$

$$\omega = -k(qr(n+1)^2+p^2)/(n+1)p \tag{15}$$

Substituting (13) – (15) into (12) and by (10) respectively we obtain:

$$u_1(x,t) = \left(\frac{k\lambda q(n+1)+np}{2np} + \frac{n^2p^2-(k\lambda q(n+1))^2}{4pqkn(n+1)(1-v)}\left(\frac{G'}{G}\right)^{-1}\right)^{\frac{1}{n}} \tag{16}$$

$$u_2(x,t) = \left(\frac{np-k\lambda q(n+1)}{2np} + \frac{kq(1-v)(n+1)}{np}\left(\frac{G'}{G}\right)\right)^{\frac{1}{n}} \tag{17}$$

$$u_3(x,t)=\left(\frac{1}{2}+\frac{kq(n+1)(1-v)}{np}\left(\frac{G'}{G}\right)+\frac{np}{16kq(n+1)(1-v)}\left(\frac{G'}{G}\right)^{-1}\right)^{\frac{1}{n}} \tag{18}$$

Substituting the solutions of the (6) (see appendix I) into (16) and simplifying obtains the following solutions of our target equation (8)

when $\Omega=\lambda^2-4\mu(v-1)>0$ *and* $\lambda(v-1)\neq 0$ (*or* $\mu(v-1)\neq 0$)

$$u_{1_1}(x,t)=\left(\frac{k\lambda q(n+1)+np}{2np}+\frac{n^2p^2-(k\lambda q(n+1))^2}{4pqkn(n+1)(1-v)}\left(-\frac{1}{2(v-1)}\left(\lambda+\sqrt{\Omega}\tanh\left(\frac{1}{2}\sqrt{\Omega}\eta\right)\right)\right)^{-1}\right)^{\frac{1}{n}} \tag{19}$$

$$u_{1_2}(x,t)=\left(\frac{k\lambda q(n+1)+np}{2np}+\frac{n^2p^2-(k\lambda q(n+1))^2}{4pqkn(n+1)(1-v)}\left(-\frac{1}{2(v-1)}\left(\lambda+\sqrt{\Omega}\coth\left(\frac{1}{2}\sqrt{\Omega}\eta\right)\right)\right)^{-1}\right)^{\frac{1}{n}} \tag{20}$$

$$u_{1_3}(x,t)=\left(\frac{k\lambda q(n+1)+np}{2np}+\frac{n^2p^2-(k\lambda q(n+1))^2}{4pqkn(n+1)(1-v)}\left(-\frac{1}{2(v-1)}\left(\lambda+\sqrt{\Omega}\left(\tanh(\frac{1}{2}\sqrt{\Omega}\eta)\pm i\,\mathrm{sech}(\frac{1}{2}\sqrt{\Omega}\eta)\right)\right)\right)^{-1}\right)^{\frac{1}{n}} \tag{21}$$

$$u_{1_4}(x,t)=\left(\frac{k\lambda q(n+1)+np}{2np}+\frac{n^2p^2-(k\lambda q(n+1))^2}{4pqkn(n+1)(1-v)}\left(-\frac{1}{2(v-1)}\left(\lambda+\sqrt{\Omega}\left(\coth(\frac{1}{2}\sqrt{\Omega}\eta)\pm \mathrm{csch}(\frac{1}{2}\sqrt{\Omega}\eta)\right)\right)\right)^{-1}\right)^{\frac{1}{n}} \tag{22}$$

$$u_{1_5}(x,t)=\left(\frac{k\lambda q(n+1)+np}{2np}+\frac{n^2p^2-(k\lambda q(n+1))^2}{4pqkn(n+1)(1-v)}\left(-\frac{1}{4(v-1)}\left(2\lambda+\sqrt{\Omega}\left(\tanh(\frac{1}{4}\sqrt{\Omega}\eta)+\coth(\frac{1}{4}\sqrt{\Omega}\eta)\right)\right)\right)^{-1}\right)^{\frac{1}{n}} \tag{23}$$

$$u_{1_6}(x,t)=\left(\frac{k\lambda q(n+1)+np}{2np}+\frac{n^2p^2-(k\lambda q(n+1))^2}{4pqkn(n+1)(1-v)}\left(\frac{1}{2(v-1)}\left(-\lambda+\frac{\pm\sqrt{\Omega(A^2-B^2)}-A\sqrt{\Omega}\cosh(\sqrt{\Omega}\eta)}{A\sinh(\sqrt{\Omega}\eta)+\mathrm{B}}\right)\right)^{-1}\right)^{\frac{1}{n}} \tag{24}$$

$$u_{1_7}(x,t)=\left(\alpha_0+\frac{n^2p^2-(k\lambda q(n+1))^2}{4pqkn(n+1)(1-v)}\left(\frac{1}{2(v-1)}\left(-\lambda+\frac{\pm\sqrt{\Omega(A^2+B^2)}-A\sqrt{\Omega}\cosh(\sqrt{\Omega}\eta)}{A\sinh(\sqrt{\Omega}\eta)+\mathrm{B}}\right)\right)^{-1}\right)^{\frac{1}{n}} \tag{25}$$

where A and B are two non-zero real constants

$$u_{1_8}(x,t)=\left(\frac{k\lambda q(n+1)+np}{2np}+\frac{n^2p^2-(k\lambda q(n+1))^2}{4pqkn(n+1)(1-v)}\left(\frac{-2\mu\sqrt{\Omega}\cosh(1/2\sqrt{\Omega})}{\sqrt{\Omega}\sinh(\sqrt{\Omega})+\lambda\cosh(1/2\sqrt{\Omega})}\right)^{-1}\right)^{\frac{1}{n}} \tag{26}$$

$$u_{1_9}(x,t)=\left(\frac{k\lambda q(n+1)+np}{2np}+\frac{n^2p^2-(k\lambda q(n+1))^2}{4pqkn(n+1)(1-v)}\left(\frac{2\mu\sqrt{\Omega}\sinh\left((1/2)\sqrt{\Omega}\right)}{\sqrt{\Omega}\cosh(\sqrt{\Omega})-\lambda\sinh\left((1/2)\sqrt{\Omega}\right)}\right)^{-1}\right)^{\frac{1}{n}} \tag{27}$$

$$u_{1_{10}}(x,t)=\left(\frac{k\lambda q(n+1)+np}{2np}+\frac{n^2p^2-(k\lambda q(n+1))^2}{4pqkn(n+1)(1-v)}\left(\frac{-2\mu\sqrt{\Omega}\cosh\left((1/2)\sqrt{\Omega}\right)}{\sqrt{\Omega}\sin(\sqrt{\Omega})+\lambda\cosh\left((1/2)\sqrt{\Omega}\right)\pm i\sqrt{\Omega}}\right)^{-1}\right)^{\frac{1}{n}} \tag{28}$$

$$u_{1_{11}}(x,t)=\left(\frac{k\lambda q(n+1)+np}{2np}+\frac{n^2p^2-(k\lambda q(n+1))^2}{4pqkn(n+1)(1-v)}\left(\frac{2\mu\sqrt{\Omega}\sinh\left((1/2)\sqrt{\Omega}\right)}{\sqrt{\Omega}\cosh(\sqrt{\Omega})-\lambda\sinh\left((1/2)\sqrt{\Omega}\right)\pm\sqrt{\Omega}}\right)^{-1}\right)^{\frac{1}{n}} \tag{29}$$

when $\Omega = \lambda^2-4\mu(v-1)<0$ *and* $\lambda(v-1) \neq 0$ (*or* $\mu(v-1) \neq 0$)

$$u_{1_{12}}(x,t)=\left(\frac{k\lambda q(n+1)+np}{2np}+\frac{n^2p^2-(k\lambda q(n+1))^2}{4pqkn(n+1)(1-v)}\left(\frac{1}{2(v-1)}\left(-\lambda+\sqrt{-\Omega}\tan\left(\frac{1}{2}\sqrt{-\Omega}\eta\right)\right)\right)^{-1}\right)^{\frac{1}{n}} \tag{30}$$

$$u_{1_{13}}(x,t)=\left(\frac{k\lambda q(n+1)+np}{2np}+\frac{n^2p^2-(k\lambda q(n+1))^2}{4pqkn(n+1)(1-v)}\left(\frac{1}{2(v-1)}\left(-\lambda+\sqrt{-\Omega}\cot\left(\frac{1}{2}\sqrt{-\Omega}\eta\right)\right)\right)^{-1}\right)^{\frac{1}{n}} \tag{31}$$

$$u_{1_{14}}(x,t)=\left(\frac{k\lambda q(n+1)+np}{2np}+\frac{n^2p^2-(k\lambda q(n+1))^2}{4pqkn(n+1)(1-v)}\left(\frac{1}{2(v-1)}\left(-\lambda+\sqrt{-\Omega}\left(\tan(\frac{1}{2}\sqrt{-\Omega}\eta)\pm\sec(\frac{1}{2}\sqrt{-\Omega}\eta)\right)\right)\right)^{-1}\right)^{\frac{1}{n}} \tag{32}$$

$$u_{15}(x,t)=\left(\frac{k\lambda q(n+1)+np}{2np}+\frac{n^2p^2-(k\lambda q(n+1))^2}{4pqkn(n+1)(1-v)}\left(\frac{1}{2(v-1)}\left(\lambda+\sqrt{-\Omega}\left(\cot(\frac{1}{2}\sqrt{-\Omega}\eta)\pm csc(\frac{1}{2}\sqrt{-\Omega}\eta)\right)\right)\right)\right)^{\frac{1}{n}} \tag{33}$$

$$u_{1_{16}}(x,t)=\left(\frac{k\lambda q(n+1)+np}{2np}+\frac{n^2p^2-(k\lambda q(n+1))^2}{4pqkn(n+1)(1-v)}\left(\frac{1}{4(v-1)}\left(-2\lambda+\sqrt{-\Omega}\left(\tan(\frac{1}{4}\sqrt{-\Omega}\eta)+\cot(\frac{1}{4}\sqrt{-\Omega}\eta)\right)\right)\right)^{-1}\right)^{\frac{1}{n}} \tag{34}$$

$$u_{1_{17}}(x,t)=\left(\frac{k\lambda q(n+1)+np}{2np}+\frac{n^2p^2-(k\lambda q(n+1))^2}{4pqkn(n+1)(1-v)}\left(\frac{1}{2(v-1)}\left(-\lambda+\frac{\pm\sqrt{-\Omega(A^2-B^2)}-A\sqrt{-\Omega}\cos(\sqrt{-\Omega}\eta)}{A\sin(\sqrt{-\Omega}\eta)+\mathrm{B}}\right)\right)^{-1}\right)^{\frac{1}{n}} \tag{35}$$

$$u_{1_{18}}(x,t)=\left(\frac{k\lambda q(n+1)+np}{2np}+\frac{n^2p^2-(k\lambda q(n+1))^2}{4pqkn(n+1)(1-v)}\left(\frac{1}{2(v-1)}\left(-\lambda+\frac{\pm\sqrt{-\Omega(A^2-B^2)}+A\sqrt{\Omega}\cos(\sqrt{-\Omega}\eta)}{A\sin(\sqrt{-\Omega}\eta)+\mathrm{B}}\right)\right)^{-1}\right)^{\frac{1}{n}} \tag{36}$$

$$u_{1_{19}}(x,t)=\left(\frac{k\lambda q(n+1)+np}{2np}+\frac{n^2p^2-(k\lambda q(n+1))^2}{4pqkn(n+1)(1-v)}\left(\frac{-2\mu\sqrt{-\Omega}\cos(1/2\sqrt{-\Omega})}{\sqrt{-\Omega}\sin(\sqrt{-\Omega})+\lambda\cos(1/2\sqrt{-\Omega})}\right)^{-1}\right)^{\frac{1}{n}} \tag{37}$$

$$u_{120}(x,t)=\left(\frac{k\lambda q(n+1)+np}{2np}+\frac{n^2p^2-(k\lambda q(n+1))^2}{4pqkn(n+1)(1-v)}\left(\frac{2\mu\sqrt{-\Omega}\sin\left((1/2)\sqrt{-\Omega}\right)}{\sqrt{-\Omega}\cos(\sqrt{-\Omega})-\lambda\sin\left((1/2)\sqrt{-\Omega}\right)}\right)^{-1}\right)^{\frac{1}{n}} \quad (38)$$

$$u_{121}(x,t)=\left(\frac{k\lambda q(n+1)+np}{2np}+\frac{n^2p^2-(k\lambda q(n+1))^2}{4pqkn(n+1)(1-v)}\left(\frac{-2\mu\sqrt{-\Omega}\cos\left((1/2)\sqrt{-\Omega}\right)}{\sqrt{-\Omega}\sin(\sqrt{-\Omega})+\lambda\cos\left((1/2)\sqrt{-\Omega}\right)\pm\sqrt{-\Omega}}\right)^{-1}\right)^{\frac{1}{n}} \quad (39)$$

$$u_{122}(x,t)=\left(\frac{k\lambda q(n+1)+np}{2np}+\frac{n^2p^2-(k\lambda q(n+1))^2}{4pqkn(n+1)(1-v)}\left(\frac{2\mu\sqrt{-\Omega}\sin\left((1/2)\sqrt{-\Omega}\right)}{\sqrt{-\Omega}\cos(\sqrt{-\Omega})-\lambda\sin\left((1/2)\sqrt{-\Omega}\right)\pm\sqrt{-\Omega}}\right)^{-1}\right)^{\frac{1}{n}} \quad (40)$$

$$u_{123}(x,t)=\left(\frac{k\lambda q(n+1)+np}{2np}+\frac{n^2p^2-(k\lambda q(n+1))^2}{4pqkn(n+1)(1-v)}\left(-\frac{\lambda k}{(v-1)(k+\cosh(\lambda\eta)-\lambda\sinh(\lambda\eta))}\right)^{-1}\right)^{\frac{1}{n}} \quad (41)$$

$$u_{124}(x,t)=\left(\frac{k\lambda q(n+1)+np}{2np}+\frac{n^2p^2-(k\lambda q(n+1))^2}{4pqkn(n+1)(1-v)}\left(-\frac{\cosh(\lambda\eta)+\lambda\sinh(\lambda\eta)}{(v-1)(k+\cosh(\lambda\eta)+\lambda\sinh(\lambda\eta))}\right)^{-1}\right)^{\frac{1}{n}} \quad (42)$$

$$u_{125}(x,t)=\left(\frac{k\lambda q(n+1)+np}{2np}+\frac{n^2p^2-(k\lambda q(n+1))^2}{4pqkn(n+1)(1-v)}\left(-\frac{1}{(v-1)\eta+1}\right)^{-1}\right)^{\frac{1}{n}} \quad (43)$$

where $\eta=kx-\frac{k(qr(n+1)^2+p^2)}{(n+1)p}t$, and k is constant in (19)-(43)

Substituting the solutions of the (6) (see appendix I) into (17) and simplifying obtain the following solutions of our target equation (8)

when $\Omega=\lambda^2-4\mu(v-1)>0$ *and* $\lambda(v-1)\neq 0$ (*or* $\mu(v-1)\neq 0$)

$$u(x,t)_{2_1}=\left(\frac{1}{2}+\frac{kq(n+1)}{2np}\sqrt{\Omega}\tanh\left(\frac{1}{2}\sqrt{\Omega}\eta\right)\right)^{\frac{1}{n}} \quad (44)$$

$$u(x,t)_{2_2}=\left(\frac{1}{2}+\frac{kq(n+1)}{2np}\sqrt{\Omega}\coth\left(\frac{1}{2}\sqrt{\Omega}\eta\right)\right)^{\frac{1}{n}} \quad (45)$$

$$u_{2_3}(x,t)=\left(\frac{1}{2}+\frac{kq(n+1)}{2np}\sqrt{\Omega}\left(\tanh(\frac{1}{2}\sqrt{\Omega}\eta)\pm i\,\mathrm{sech}(\frac{1}{2}\sqrt{\Omega}\eta)\right)\right)^{\frac{1}{n}} \quad (46)$$

$$u_{2_4}(x,t)=\left(\frac{1}{2}+\frac{kq(n+1)}{2np}\sqrt{\Omega}\left(\coth(\frac{1}{2}\sqrt{\Omega}\eta)\pm \mathrm{csch}(\frac{1}{2}\sqrt{\Omega}\eta)\right)\right)^{\frac{1}{n}} \quad (47)$$

$$u_{2_5}(x,t)=\left(\frac{1}{2}+\frac{kq(n+1)}{4np}\sqrt{\Omega}\left(\tanh(\frac{1}{4}\sqrt{\Omega}\eta)+\coth(\frac{1}{4}\sqrt{\Omega}\eta)\right)\right)^{\frac{1}{n}} \quad (48)$$

$$u_{2_6}(x,t)=\left(\frac{1}{2}-\frac{kq(n+1)}{2np}\frac{\pm\sqrt{\Omega(A^2+B^2)}-A\sqrt{\Omega}\cosh(\sqrt{\Omega}\eta)}{A\sinh(\sqrt{\Omega}\eta)+\mathrm{B}}\right)^{\frac{1}{n}} \quad (49)$$

$$u_{2_7}(x,t)=\left(\frac{1}{2}+\frac{kq(n+1)}{2np}\frac{\pm\sqrt{\Omega(A^2+B^2)}-A\sqrt{\Omega}\cosh(\sqrt{\Omega}\eta)}{A\sinh(\sqrt{\Omega}\eta)+\mathrm{B}}\right)^{\frac{1}{n}} \tag{50}$$

where A and B are two non-zero real constants

$$u_{2_8}(x,t)=\left(\frac{np-k\lambda q(n+1)}{2np}+\frac{kq(1-v)(n+1)}{np}\left(\frac{-2\mu\sqrt{\Omega}\cosh(1/2\sqrt{\Omega})}{\sqrt{\Omega}\sinh(\sqrt{\Omega})+\lambda\cosh(1/2\sqrt{\Omega})}\right)\right)^{\frac{1}{n}} \tag{51}$$

$$u_{2_9}(x,t)=\left(\frac{np-k\lambda q(n+1)}{2np}+\frac{kq(1-v)(n+1)}{np}\left(\frac{2\mu\sqrt{\Omega}\sinh\left((1/2)\sqrt{\Omega}\right)}{\sqrt{\Omega}\cosh(\sqrt{\Omega})-\lambda\sinh\left((1/2)\sqrt{\Omega}\right)}\right)\right)^{\frac{1}{n}} \tag{52}$$

$$u_{2_{10}}(x,t)=\left(\frac{np-k\lambda q(n+1)}{2np}+\frac{kq(1-v)(n+1)}{np}\left(\frac{-2\mu\sqrt{\Omega}\cosh\left((1/2)\sqrt{\Omega}\right)}{\sqrt{\Omega}\sin(\sqrt{\Omega})+\lambda\cosh\left((1/2)\sqrt{\Omega}\right)\pm i\sqrt{\Omega}}\right)\right)^{\frac{1}{n}} \tag{53}$$

$$u_{2_{11}}(x,t)=\left(\frac{np-k\lambda q(n+1)}{2np}+\frac{kq(1-v)(n+1)}{np}\left(\frac{2\mu\sqrt{\Omega}\sinh\left((1/2)\sqrt{\Omega}\right)}{\sqrt{\Omega}\cosh(\sqrt{\Omega})-\lambda\sinh\left((1/2)\sqrt{\Omega}\right)\pm\sqrt{\Omega}}\right)\right)^{\frac{1}{n}} \tag{54}$$

when $\Omega = \lambda^2-4\mu(v-1)<0$ *and* $\lambda(v-1)\neq 0$(*or* $\mu(v-1)\neq 0$)

$$u(x,t)_{2_{12}}=\left(\frac{1}{2}+\frac{kq(n+1)}{2np}\sqrt{-\Omega}\tan\left(\frac{1}{2}\sqrt{-\Omega}\eta\right)\right)^{\frac{1}{n}} \tag{55}$$

$$u(x,t)_{2_{13}}=\left(\frac{1}{2}+\frac{kq(n+1)}{2np}\sqrt{-\Omega}\cot\left(\frac{1}{2}\sqrt{-\Omega}\eta\right)\right)^{\frac{1}{n}} \tag{56}$$

Here for simplicity we omitted solutions of the form $u_{2_{14}}-u_{2_{25}}$ (of the form $u_{14}-u_{25}$ as in appendix I)

where $\eta=kx-\frac{k(qr(n+1)^2+p^2)}{(n+1)p}t$, and k is constant in (44)-(56)

Substituting the solutions of the (6) (see appendix I) into (18) and simplifying obtain the following solutions of our target equation (8)

when $\Omega = \lambda^2-4\mu(v-1)>0$ *and* $\lambda(v-1)\neq 0$(*or* $\mu(v-1)\neq 0$)

$$u_{3_1}(x,t)=\left(\begin{array}{l}\frac{1}{2}+\frac{kq(n+1)}{2np}\left(\sqrt{\Omega}\tanh\left(\frac{1}{2}\sqrt{\Omega}\eta\right)\right)+\\ \frac{np}{16kq(n+1)(1-v)}\left(-\frac{1}{2(v-1)}\left(\sqrt{\Omega}\tanh\left(\frac{1}{2}\sqrt{\Omega}\eta\right)\right)\right)^{-1}\end{array}\right)^{\frac{1}{n}} \tag{57}$$

$$u_{3_2}(x,t)=\left(\begin{array}{l}\frac{1}{2}+\frac{kq(n+1)}{2np}\left(\sqrt{\Omega}\coth\left(\frac{1}{2}\sqrt{\Omega}\eta\right)\right)+\\ \frac{np}{16kq(n+1)(1-v)}\left(-\frac{1}{2(v-1)}\left(\sqrt{\Omega}\coth\left(\frac{1}{2}\sqrt{\Omega}\eta\right)\right)\right)^{-1}\end{array}\right)^{\frac{1}{n}} \tag{58}$$

$$u_{3_3}(x,t)=\left(\begin{array}{l}\frac{1}{2}+\frac{kq(n+1)}{2np}\left(\sqrt{\Omega}\left(\tanh(\frac{1}{2}\sqrt{\Omega}\eta)\pm i\sec h(\frac{1}{2}\sqrt{\Omega}\eta)\right)\right)+\\ \frac{np}{16kq(n+1)(1-v)}\left(-\frac{1}{2(v-1)}\left(\sqrt{\Omega}\left(\tanh(\frac{1}{2}\sqrt{\Omega}\eta)\pm i\sec h(\frac{1}{2}\sqrt{\Omega}\eta)\right)\right)\right)^{-1}\end{array}\right)^{\frac{1}{n}} \tag{59}$$

$$u_{3_4}(x,t)=\left(\begin{array}{l}\frac{1}{2}+\frac{kq(n+1)}{2np}\left(\sqrt{\Omega}\left(\coth(\frac{1}{2}\sqrt{\Omega}\eta)\pm cs\,\mathrm{c}\,h(\frac{1}{2}\sqrt{\Omega}\eta)\right)\right)+\\ \frac{np}{16kq(n+1)(1-v)}\left(-\frac{1}{2(v-1)}\left(\sqrt{\Omega}\left(\coth(\frac{1}{2}\sqrt{\Omega}\eta)\pm cs\,\mathrm{c}\,h(\frac{1}{2}\sqrt{\Omega}\eta)\right)\right)\right)^{-1}\end{array}\right)^{\frac{1}{n}} \tag{60}$$

$$u_{3_5}(x,t)=\left(\begin{array}{l}\frac{1}{2}+\frac{kq(n+1)}{4np}\left(\sqrt{\Omega}\left(\tanh(\frac{1}{4}\sqrt{\Omega}\eta)+\coth(\frac{1}{4}\sqrt{\Omega}\eta)\right)\right)+\\ \frac{np}{16kq(n+1)(1-v)}\left(-\frac{1}{4(v-1)}\left(\sqrt{\Omega}\left(\tanh(\frac{1}{4}\sqrt{\Omega}\eta)+\coth(\frac{1}{4}\sqrt{\Omega}\eta)\right)\right)\right)^{-1}\end{array}\right)^{\frac{1}{n}} \tag{61}$$

$$u_{3_6}(x,t)=\left(\begin{array}{l}\frac{1}{2}-\frac{kq(n+1)}{2np}\left(\frac{\pm\sqrt{\Omega(A^2+B^2)}-A\sqrt{\Omega}\cosh(\sqrt{\Omega}\eta)}{A\sinh(\sqrt{\Omega}\eta)+\mathrm{B}}\right)+\\ \frac{np}{16kq(n+1)(1-v)}\left(\frac{1}{2(v-1)}\left(\frac{\pm\sqrt{\Omega(A^2+B^2)}-A\sqrt{\Omega}\cosh(\sqrt{\Omega}\eta)}{A\sinh(\sqrt{\Omega}\eta)+\mathrm{B}}\right)\right)^{-1}\end{array}\right)^{\frac{1}{n}} \tag{62}$$

$$u_{3_7}(x,t)=\left(\begin{array}{l}\frac{1}{2}-\frac{kq(n+1)}{2np}\left(-\frac{\pm\sqrt{\Omega(A^2+B^2)}+A\sqrt{\Omega}\cosh(\sqrt{\Omega}\eta)}{A\sinh(\sqrt{\Omega}\eta)+\mathrm{B}}\right)+\\ \frac{np}{16kq(n+1)(1-v)}\left(\frac{1}{2(v-1)}\left(-\frac{\pm\sqrt{\Omega(A+B^2)}+A\sqrt{\Omega}\cosh(\sqrt{\Omega}\eta)}{A\sinh(\sqrt{\Omega}\eta)+\mathrm{B}}\right)\right)^{-1}\end{array}\right)^{\frac{1}{n}} \tag{63}$$

where A and B are two non-zero real constants

$$u(x,t)_{3_8}=\left(\begin{array}{l}\frac{1}{2}+\frac{kq(n+1)(1-v)}{np}\left(\frac{-2\mu\sqrt{\Omega}\cosh(1/2\sqrt{\Omega})}{\sqrt{\Omega}\sinh(\sqrt{\Omega})+\lambda\cosh(1/2\sqrt{\Omega})}\right)+\\ \frac{np}{16kq(n+1)(1-v)}\left(\frac{-2\mu\sqrt{\Omega}\cosh(1/2\sqrt{\Omega})}{\sqrt{\Omega}\sinh(\sqrt{\Omega})+\lambda\cosh(1/2\sqrt{\Omega})}\right)^{-1}\end{array}\right)^{\frac{1}{n}} \tag{64}$$

$$u_{3_9}(x,t)=\left(\begin{array}{l}\frac{1}{2}+\frac{kq(n+1)(1-v)}{np}\left(\frac{2\mu\sqrt{\Omega}\sinh\left((1/2)\sqrt{\Omega}\right)}{\sqrt{\Omega}\cosh(\sqrt{\Omega})-\lambda\sinh\left((1/2)\sqrt{\Omega}\right)}\right)+\\ \frac{np}{16kq(n+1)(1-v)}\left(\frac{2\mu\sqrt{\Omega}\sinh\left((1/2)\sqrt{\Omega}\right)}{\sqrt{\Omega}\cosh(\sqrt{\Omega})-\lambda\sinh\left((1/2)\sqrt{\Omega}\right)}\right)^{-1}\end{array}\right)^{\frac{1}{n}} \quad (65)$$

$$u_{3_{10}}(x,t)=\left(\begin{array}{l}\frac{1}{2}+\frac{kq(n+1)(1-v)}{np}\left(\frac{-2\mu\sqrt{\Omega}\cosh\left((1/2)\sqrt{\Omega}\right)}{\sqrt{\Omega}\sin(\sqrt{\Omega})+\lambda\cosh\left((1/2)\sqrt{\Omega}\right)\pm i\sqrt{\Omega}}\right)+\\ \frac{np}{16kq(n+1)(1-v)}\left(\frac{-2\mu\sqrt{\Omega}\cosh\left((1/2)\sqrt{\Omega}\right)}{\sqrt{\Omega}\sin(\sqrt{\Omega})+\lambda\cosh\left((1/2)\sqrt{\Omega}\right)\pm i\sqrt{\Omega}}\right)^{-1}\end{array}\right)^{\frac{1}{n}} \quad (66)$$

$$u_{3_{11}}(x,t)=\left(\begin{array}{l}\frac{1}{2}+\frac{kq(n+1)(1-v)}{np}\left(\frac{2\mu\sqrt{\Omega}\sinh\left((1/2)\sqrt{\Omega}\right)}{\sqrt{\Omega}\cosh(\sqrt{\Omega})-\lambda\sinh\left((1/2)\sqrt{\Omega}\right)\pm\sqrt{\Omega}}\right)+\\ \frac{np}{16kq(n+1)(1-v)}\left(\frac{2\mu\sqrt{\Omega}\sinh\left((1/2)\sqrt{\Omega}\right)}{\sqrt{\Omega}\cosh(\sqrt{\Omega})-\lambda\sinh\left((1/2)\sqrt{\Omega}\right)\pm\sqrt{\Omega}}\right)^{-1}\end{array}\right)^{\frac{1}{n}} \quad (67)$$

when $\Omega = \lambda^2-4\mu(v-1) > 0$ *and* $\lambda(v-1) \neq 0$ (*or* $\mu(v-1) \neq 0$)

$$u_{3_{12}}(x,t)=\left(\begin{array}{l}\frac{1}{2}+\frac{kq(n+1)}{2np}\left(-\lambda+\sqrt{-\Omega}\tan\left(\frac{1}{2}\sqrt{-\Omega}\eta\right)\right)+\\ \frac{np}{16kq(n+1)(1-v)}\left(\frac{1}{2(v-1)}\left(-\lambda+\sqrt{-\Omega}\tan\left(\frac{1}{2}\sqrt{-\Omega}\eta\right)\right)\right)^{-1}\end{array}\right)^{\frac{1}{n}} \quad (68)$$

$$u_{3_{13}}(x,t)=\left(\begin{array}{l}\frac{1}{2}+\frac{kq(n+1)}{2np}\left(-\lambda+\sqrt{-\Omega}\cot\left(\frac{1}{2}\sqrt{-\Omega}\eta\right)\right)+\\ \frac{np}{16kq(n+1)(1-v)}\left(\frac{1}{2(v-1)}\left(-\lambda+\sqrt{-\Omega}\cot\left(\frac{1}{2}\sqrt{-\Omega}\eta\right)\right)\right)^{-1}\end{array}\right)^{\frac{1}{n}} \quad (69)$$

Here for simplicity we omitted solutions of the form $u_{3_{14}}-u_{3_{25}}$ (of the form $u_{14}-u_{25}$ as in appendix I)

where $\eta = kx-\frac{k(qr(n+1)^2+p^2)}{(n+1)p}t$, and k is constant in (57)-(69)

4. Results and Discussion

Recently, Abaker A. Hassaballa et al [23] applied the proposed method, improved (G'/G) -expansion method to Burger-fisher equation which is special the case of our target equation where $G=G(\eta)$ satisfies $G''+\lambda G'+\mu G=0$ and obtained five exact traveling wave solutions in terms of hyperbolic functions. Another authors, Abdollah Boharfani et al [22] applied the basic (G'/G) -expansion method to generalized Burger-fisher equation which is our target equation where $G=G(\eta)$ satisfies $G''+\lambda G'+\mu G=0$ and found a few exact traveling wave solutions as compared to [23] and our solutions.

In our case, we applied the improved (G'/G) -expansion method to the generalized Burger-fisher equation where $G=G(\eta)$ satisfies $GG''=\lambda GG'+\mu G^2+v(G')^2$ as a result we have constructed more general solutions and many new exact traveling solutions.

It is important to point out that some of constructed solutions are in good agreement with already published results which have been shown in the Table 1.

Table 1. *Comparison between published results and our obtained solutions.*

1) Abaker A. Hassaballa et al Solutions [23]	Our Solutions
i). Equation (18) with $\xi = x - \frac{(4\beta+\alpha^2)}{2\alpha}t$ & $\psi=\lambda^2 + \frac{(\alpha-2\lambda)(\alpha+2\lambda)}{4} = \frac{\alpha^2}{4}$ $u_{1_1} = \left(\frac{1}{2}\left(1-\frac{2}{\alpha}\sqrt{\psi}\tanh\left(\frac{1}{2}\sqrt{\psi}\xi\right)\right)\right)$ or $u_{1_1} = \left(\frac{1}{2}\left(1-\frac{1}{\alpha}\sqrt{\alpha^2}\tanh\left(\frac{1}{2}\sqrt{\alpha^2}\xi\right)\right)\right)$	i). Equation (44) and (14) with $n=1, k=1, p=\alpha, r=-\beta, q=-1$ and $v=0$ $\xi=\eta=kx-\frac{k(qr(n+1)^2+p^2)}{p(n+1)}t = x-\frac{(4\beta^2+\alpha^2)}{2\alpha}t$ $u_{2_2} = \left(\frac{1}{2}\left(-\frac{1}{\alpha}\sqrt{\alpha^2}\tanh\left(\frac{1}{4}\sqrt{\alpha^2}\eta\right)\right)\right)$
ii). Equation (19) with $\xi = x - \frac{(4\beta+\alpha^2)}{2\alpha}t$ & $\psi=\lambda^2 + \frac{(\alpha-2\lambda)(\alpha+2\lambda)}{4} = \frac{\alpha^2}{4}$ $u_{1_2} = \left(\frac{1}{2}\left(1-\frac{2}{\alpha}\sqrt{\psi}\coth\left(\frac{1}{2}\sqrt{\psi}\xi\right)\right)\right)$ or $u_{1_2} = \left(\frac{1}{2}\left(1-\frac{1}{\alpha}\sqrt{\alpha^2}\coth\left(\frac{1}{2}\sqrt{\alpha^2}\xi\right)\right)\right)$	ii). Equation (45) and (14) with $n=1, k=1, p=\alpha, r=-\beta, q=-1$ and $v=0$ $\xi=\eta=kx-\frac{k(qr(n+1)^2+p^2)}{p(n+1)}t = x-\frac{(4\beta^2+\alpha^2)}{2\alpha}t$ $u_{2_2} = \left(\frac{1}{2}\left(-\frac{1}{\alpha}\sqrt{\alpha^2}\coth\left(\frac{1}{4}\sqrt{\alpha^2}\eta\right)\right)\right)$
iii) Equation (20) with $\xi = x - \frac{(4\beta+\alpha^2)}{2\alpha}t$ & $\psi=\lambda^2 + \frac{(\alpha-2\lambda)(\alpha+2\lambda)}{4} = \frac{\alpha^2}{4}$ $u_{2_1}(x,t) = \frac{\alpha+2\lambda}{2\alpha}\left(1-\left(\frac{2\sqrt{\psi}}{\alpha-2\lambda}\tanh\left(\frac{1}{2}\sqrt{\psi^2}\eta\right)-\frac{2\lambda}{\alpha-2\lambda}\right)^{-1}\right)$ or $u_{2_1}(x,t) = \frac{\alpha+2\lambda}{2\alpha}\left(1-\left(\frac{\sqrt{\alpha^2}}{\alpha-2\lambda}\tanh\left(\frac{1}{4}\sqrt{\alpha^2}\eta\right)-\frac{2\lambda}{\alpha-2\lambda}\right)^{-1}\right)$	iii) Equation (19) and (13) with $n=1, k=1, p=\alpha, r=-\beta, q=-1, \lambda$ by $-\lambda$ and $v=0$ $\xi=\eta=kx-\frac{k(qr(n+1)^2+p^2)}{p(n+1)}t = x-\frac{(4\beta^2+\alpha^2)}{2\alpha}t$ $u_{1_1}(x,t) = \frac{\alpha+2\lambda}{2\alpha}-\frac{\alpha^2-4\lambda^2}{8\alpha}\left(-\frac{\lambda}{2}+\frac{\sqrt{\alpha^2}}{4}\tanh\left(\frac{1}{4}\sqrt{\alpha^2}\eta\right)\right)^{-1}$ or $u_{1_1}(x,t) = \frac{\alpha+2\lambda}{2\alpha}\left(1-\left(\frac{\sqrt{\alpha^2}}{\alpha-2\lambda}\tanh\left(\frac{1}{4}\sqrt{\alpha^2}\eta\right)-\frac{2\lambda}{\alpha-2\lambda}\right)^{-1}\right)$
iv) Equation (21) with $\xi = x - \frac{(4\beta+\alpha^2)}{2\alpha}t$ & $\psi=\lambda^2 + \frac{(\alpha-2\lambda)(\alpha+2\lambda)}{4} = \frac{\alpha^2}{4}$ $u_{2_2}(x,t) = \frac{\alpha+2\lambda}{2\alpha}\left(1-\left(\frac{2\sqrt{\psi}}{\alpha-2\lambda}\coth\left(\frac{1}{2}\sqrt{\psi^2}\eta\right)-\frac{2\lambda}{\alpha-2\lambda}\right)^{-1}\right)$ or $= \frac{\alpha+2\lambda}{2\alpha}\left(1-\left(\frac{\sqrt{\alpha^2}}{\alpha-2\lambda}\coth\left(\frac{1}{4}\sqrt{\alpha^2}\eta\right)-\frac{2\lambda}{\alpha-2\lambda}\right)^{-1}\right)$	iv) Equation (20) and (13) with $n=1, k=1, p=\alpha, r=-\beta, q=-1, \lambda$ by $-\lambda$ and $v=0$ $\xi=\eta=kx-\frac{k(qr(n+1)^2+p^2)}{p(n+1)}t = x-\frac{(4\beta^2+\alpha^2)}{2\alpha}t$ $u_{1_2}(x,t) = \frac{\alpha+2\lambda}{2\alpha}-\frac{\alpha^2-4\lambda^2}{8\alpha}\left(-\frac{\lambda}{2}+\frac{\sqrt{\alpha^2}}{4}\coth\left(\frac{1}{4}\sqrt{\alpha^2}\eta\right)\right)^{-1}$ or $u_{1_2}(x,t) = \frac{\alpha+2\lambda}{2\alpha}\left(1-\left(\frac{\sqrt{\alpha^2}}{\alpha-2\lambda}\coth\left(\frac{1}{4}\sqrt{\alpha^2}\eta\right)-\frac{2\lambda}{\alpha-2\lambda}\right)^{-1}\right)$
v). Equation (22) with	i). Equation (57) and (15) with

$\xi = x - \frac{(4\beta+\alpha^2)}{2\alpha}t$ & $\psi=\lambda^2 + \frac{(\alpha-2\lambda)(\alpha+2\lambda)}{4} = \frac{\alpha^2}{4}$ $u_3(x,t) = \frac{1}{2} - \frac{1}{4}\tanh\left(\frac{\alpha}{8}\xi\right) - \frac{1}{4}\coth\left(\frac{\alpha}{8}\xi\right)$	$n=1, k=1, p=\alpha, r=-\beta, q=-1$ and $v=0$ $\xi = \eta = kx - \frac{k(qr(n+1)^2+p^2)}{p(n+1)}t = x - \frac{(4\beta+\alpha^2)}{2\alpha}t$ $u_{3_1}(x,t) = \left(\frac{1}{2} - \frac{1}{4}\tanh\left(\frac{\alpha}{8}\eta\right) - \frac{\alpha}{32}\left(\frac{1}{2}\left(\frac{\alpha}{4}\tanh\left(\frac{\alpha}{8}\eta\right)\right)\right)^{-1}\right)$ $= \frac{1}{2} - \frac{1}{4}\tanh\left(\frac{\alpha}{8}\xi\right) - \frac{1}{4}\coth\left(\frac{\alpha}{8}\xi\right)$
2) Abdollah Boharfani et al Solutions [22]	Our Solutions
i). Equation (34) with $\xi = kx - \frac{k(qr(n+1)^2+p^2)}{p(n+1)}t$ $u_{1,2} = \left[\left(\frac{qk}{2p}\sqrt{\frac{p^2}{q^2k^2}}\coth\left(\frac{n}{2(n+1)}\sqrt{\frac{p^2}{q^2k^2}}\xi\right)\right) + \frac{1}{2}\right]^{\frac{1}{n}}$	i). Equation (45) and (14) with $\xi = \eta = kx - \frac{k(qr(n+1)^2+p^2)}{p(n+1)}t$ $u_{2_1} = \left[\frac{1}{2} + \frac{qk}{2p}\sqrt{\frac{p^2}{q^2k^2}}\coth\left(\frac{n}{2(n+1)}\sqrt{\frac{p^2}{q^2k^2}}\eta\right)\right]^{\frac{1}{n}}$
iii) Equation (32) with $\xi = kx - \frac{k(qr(n+1)^2+p^2)}{p(n+1)}t$ $u_{1,1} = \left[\frac{1}{2} + \frac{qk}{2p}\sqrt{\frac{p^2}{q^2k^2}}\tanh\left(\frac{n}{2(n+1)}\sqrt{\frac{p^2}{q^2k^2}}\xi\right)\right]^{\frac{1}{n}}$	iii) Equation (44) and (14) with $\xi = \eta = kx - \frac{k(qr(n+1)^2+p^2)}{p(n+1)}t$ $u_{2_1} = \left[\frac{1}{2} + \frac{qk}{2p}\sqrt{\frac{p^2}{q^2k^2}}\tanh\left(\frac{n}{2(n+1)}\sqrt{\frac{p^2}{q^2k^2}}\eta\right)\right]^{\frac{1}{n}}$

Beside the above table, we found some new exact traveling wave solutions, such as, $u1_3 - u1_{11}$, $u1_{14} - u1_{25}$, $u2_3 - u2_{11}$, $u2_{14} - u2_{25}$, $u3_3 - u3_{11}$, $u3_{14} - u3_{25}$ which are not being revealed in the previous literatures.

5. Conclusions

In this article, we apply the improved (G'/G)-expansion method where G satisfies the second-order nonlinear ordinary differential equation to generate more general solution and a rich class of new exact traveling wave solutions of nonlinear PDE, namely, the generalized Burgers-Fisher equation. As result we obtained more general solution and many new exact traveling wave solutions compared to the result obtained by the improved (G'/G)-expansion method and the basic (G'/G)-expansion method where G satisfies the second-order linear ordinary differential equation. The presented solutions may express a variety of new features of waves. Moreover, the obtained exact solutions reveal that the improved (G'/G)-expansion method with the second-order nonlinear ordinary differential equation is a promising mathematical tool, because, it can establish abundant new traveling wave solutions with different physical structures. Subsequently, the used method could lead to construct many new traveling wave solutions for various nonlinear PDEs which frequently arise in scientific real time application fields.

Appendix

when $\Omega = \lambda^2 - 4\mu(v-1) > 0$ *and* $\lambda(v-1) \neq 0$ (*or* $\mu(v-1) \neq 0$)

$$u_1(x,t) = -\frac{1}{2(v-1)}\left(\lambda + \sqrt{\Omega}\tanh\left(\frac{1}{2}\sqrt{\Omega}\eta\right)\right)$$

$$u_2(x,t) = -\frac{1}{2(v-1)}\left(\lambda + \sqrt{\Omega}\coth\left(\frac{1}{2}\sqrt{\Omega}\eta\right)\right)$$

$$u_3(x,t) = -\frac{1}{2(v-1)}\left(\lambda + \sqrt{\Omega}\left(\tanh(\frac{1}{2}\sqrt{\Omega}\eta) \pm i\,\mathrm{sech}(\frac{1}{2}\sqrt{\Omega}\eta)\right)\right)$$

$$u_4(x,t) = -\frac{1}{2(v-1)}\left(\lambda + \sqrt{\Omega}\left(\coth(\frac{1}{2}\sqrt{\Omega}\eta) \pm \mathrm{csch}(\frac{1}{2}\sqrt{\Omega}\eta)\right)\right)$$

$$u_5(x,t)=\left(-\frac{1}{4(v-1)}\left(2\lambda+\sqrt{\Omega}\left(\tanh(\frac{1}{4}\sqrt{\Omega}\eta)+\coth(\frac{1}{4}\sqrt{\Omega}\eta)\right)\right)\right)$$

$$u_6(x,t)=\frac{1}{2(v-1)}\left(-\lambda+\frac{\pm\sqrt{\Omega(A^2+B^2)}-A\sqrt{\Omega}\cosh(\sqrt{\Omega}\eta)}{A\sinh(\sqrt{\Omega}\eta)+\mathrm{B}}\right)$$

$$u_7(x,t)=\frac{1}{2(v-1)}\left(-\lambda-\frac{\pm\sqrt{\Omega(A^2+B^2)}+A\sqrt{\Omega}\cosh(\sqrt{\Omega}\eta)}{A\sinh(\sqrt{\Omega}\eta)+\mathrm{B}}\right)$$

where A and B are non-zero constants.

$$u_8(x,t)=\frac{-2\mu\sqrt{\Omega}\cosh(1/2\sqrt{\Omega})}{\sqrt{\Omega}\sinh(\sqrt{\Omega})+\lambda\cosh(1/2\sqrt{\Omega})}$$

$$u_9(x,t)=\frac{2\mu\sqrt{\Omega}\sinh\left((1/2)\sqrt{\Omega}\right)}{\sqrt{\Omega}\cosh(\sqrt{\Omega})-\lambda\sinh\left((1/2)\sqrt{\Omega}\right)}$$

$$u_{10}(x,t)=\frac{-2\mu\sqrt{\Omega}\cosh\left((1/2)\sqrt{\Omega}\right)}{\sqrt{\Omega}\sin(\sqrt{\Omega})+\lambda\cosh\left((1/2)\sqrt{\Omega}\right)\pm i\sqrt{\Omega}}$$

$$u_{11}(x,t)=\frac{2\mu\sqrt{\Omega}\sinh\left((1/2)\sqrt{\Omega}\right)}{\sqrt{\Omega}\cosh(\sqrt{\Omega})-\lambda\sinh\left((1/2)\sqrt{\Omega}\right)\pm\sqrt{\Omega}}$$

when $\Omega = \lambda^2-4\mu(v-1)<0$ *and* $\lambda(v-1)\neq 0$ (*or* $\mu(v-1)\neq 0$)

$$u_{12}(x,t)=\frac{1}{2(v-1)}\left(-\lambda+\sqrt{-\Omega}\tan\left(\frac{1}{2}\sqrt{-\Omega}\eta\right)\right)$$

$$u_{13}(x,t)=\frac{1}{2(v-1)}\left(-\lambda+\sqrt{-\Omega}\cot\left(\frac{1}{2}\sqrt{-\Omega}\eta\right)\right)$$

$$u_{14}(x,t)=\frac{1}{2(v-1)}\left(-\lambda+\sqrt{-\Omega}\left(\tan(\frac{1}{2}\sqrt{-\Omega}\eta)\pm\sec(\frac{1}{2}\sqrt{-\Omega}\eta)\right)\right)$$

$$u_{15}(x,t)=-\frac{1}{2(v-1)}\left(\lambda+\sqrt{-\Omega}\left(\cot(\frac{1}{2}\sqrt{-\Omega}\eta)\pm csc(\frac{1}{2}\sqrt{-\Omega}\eta)\right)\right)$$

$$u_{16}(x,t)=\frac{1}{4(v-1)}\left(-2\lambda+\sqrt{-\Omega}\left(\tan(\frac{1}{4}\sqrt{-\Omega}\eta)+\cot(\frac{1}{4}\sqrt{-\Omega}\eta)\right)\right)$$

$$u_{17}(x,t)=\frac{1}{2(v-1)}\left(-\lambda+\frac{\pm\sqrt{-\Omega(A^2-B^2)}-A\sqrt{-\Omega}\cos(\sqrt{-\Omega}\eta)}{A\sin(\sqrt{-\Omega}\eta)+\mathrm{B}}\right)$$

$$u_{18}(x,t)=\frac{1}{2(v-1)}\left(-\lambda-\frac{\pm\sqrt{-\Omega(A^2-B^2)}+A\sqrt{\Omega}\cos(\sqrt{-\Omega}\eta)}{A\sin(\sqrt{-\Omega}\eta)+\mathrm{B}}\right)$$

where A and B are non-zero constants and satisfy the condition $A^2-B^2>0$

$$u_{19}(x,t)=\frac{-2\mu\sqrt{-\Omega}\cos(1/2\sqrt{-\Omega})}{\sqrt{-\Omega}\sin(\sqrt{-\Omega})+\lambda\cos(1/2\sqrt{-\Omega})}$$

$$u_{20}(x,t)=\frac{2\mu\sqrt{-\Omega}\sin\left((1/2)\sqrt{-\Omega}\right)}{\sqrt{-\Omega}\cos(\sqrt{-\Omega})-\lambda\sin\left((1/2)\sqrt{-\Omega}\right)}$$

$$u_{21}(x,t)=\frac{-2\mu\sqrt{-\Omega}\cos\left((1/2)\sqrt{-\Omega}\right)}{\sqrt{-\Omega}\sin(\sqrt{-\Omega})+\lambda\cos\left((1/2)\sqrt{-\Omega}\right)\pm\sqrt{-\Omega}}$$

$$u_{22}(x,t)=\frac{2\mu\sqrt{-\Omega}\sin\left((1/2)\sqrt{-\Omega}\right)}{\sqrt{-\Omega}\cos(\sqrt{-\Omega})-\lambda\sin\left((1/2)\sqrt{-\Omega}\right)\pm\sqrt{-\Omega}}$$

when $\mu=0$ and $\lambda(v-1)\neq 0$

$$u_{23}(x,t)=-\frac{\lambda k}{(v-1)(k+\cosh(\lambda\eta)-\lambda\sinh(\lambda\eta))}$$

$$u_{24}(x,t)=-\frac{\cosh(\lambda\eta)+\lambda\sinh(\lambda\eta)}{(v-1)(k+\cosh(\lambda\eta)+\lambda\sinh(\lambda\eta))}$$

where k is an arbitrary constant.
when $(v-1)\neq 0$ and $\mu=\lambda=0$

$$u_{25}(x,t)=-\frac{1}{(v-1)\eta+c_1}$$

References

[1] Bhajan lal, Exact Solutions of some Nonlinear Partial Differential Equation, Patiala- 147004 (Punjab) July-2011.

[2] J. H. He, X. H. Wu, Exp-function method for nonlinear wave equations, Chaos Solitons Fract. 30 (2006)700-708.

[3] Zhou, X. W., Wen, Y. X., He J. H.: Exp-function method to solve the nonlinear dispersive k (m, n) equations, Int. J. Non-linear Sci. Numer. Simul. 9, 301–306 (2008).

[4] Wazwaz, A. M. (2006). The tanh and the sine–cosine methods for a reliable treatment of the modified equal width equation and its variants, Communication Nonlinear Science Numerical Simulation, 11: pp. 148–60.

[5] A. M. Wazwaz, The tanh-coth method for new compactons and solitons solutions for the K (n, n) and the K (n + 1, n + 1) equations, Appl. Math. Comput., 188 (2007), 1930-1940.

[6] Nassar Hassan Abdel-All, Mohamed Abd-Allah Abdel-Razek and Abd-Allah Kamel Seddeek Expanding the Tanh-Function Method for Solving Nonlinear Equations, Applied Mathematics, 2 (2011), 1096-1104 doi: 10.4236.

[7] Y. L. Ma, B. Q. Li, A series of abundant exact travelling wave solutionsfor a modified generalized Vakhnenko equation using auxiliary equationmethod, Appl. Math. Comput., 211 (2009), 102-107.

[8] Hossein Jafari, Rahmat Soltani, Chaudry Masood Khalique and Dumitru Baleanu, Exact solutions of two nonlinear partial differential equation by using the first integral method, http://www.boundaryvalueproblems.com/content/2013/1/117.

[9] T. A. Abassy, Improved Adomian decomposition method, *Comput. Math. Appl.,* Vol. 59, No. 1, 2010, pp. 42–54.

[10] J. H. He, Variational iteration method for delay differential equations, *Commun. Nonlinear Sci. Numer. Simul.* Vol. 2, No. 4, 1997, pp. 235–236.

[11] Wang Y C, Wang L X, Zhang W B: Application of the Adomian Decomposition Method to Full Nonlinear Sine-Gordon Equation. *International Journal of Nonlinear Science.* 2 (1): 29-38 (2006).

[12] Wang, M., Li, X. and Zhang, J., "The (G'/G)-expansion method and travelling wave solutions of nonlinear evolution equations in mathematical physics," *Phys. Lett. A,* 372, 417-423, 2008.

[13] Ozis, T. and Aslan, I., "Application of the (G'/G)-expansion method to Kawahara type equations using symbolic computation," *Appl. Math. Computation*, 216, 2360-2365, 2010.

[14] Naher, H., Abdullah, F. A. and Akbar, M. A., "The (G'/G)-expansion method for abundant traveling wave solutions of Caudrey-Dodd-Gibbon equation," *Math. Prob. Eng.*, Article ID: 218216, 11 pages, 2011.

[15] Jabbari, A., Kheiri, H. and Bekir, A., "Exact solutions of the coupled Higgs equation and the Miccari system using He's semi-inverse method and (G'/G) -expansion method," *Computers Math. Appli*., 62, 2177-2186, 2011.

[16] Naher, H. and Abdullah, F. A., "The basic (G'/G)-expansion method for the fourth order Boussinesq equation," Appl. Math., 3, 1144-1152, 2012.

[17] Zhang, J. Jiang, F. and Zhao, X., "An improved (G'/G)-expansion method for solving nonlinear evolution equations," Int. J. Computer Math., 87, 1716-1725, 2010.

[18] Zhao, Y. M., Yang, Y. J. and Li, W., "Application of the improved (G'/G)-expansion method for the Variant Boussinesq equations," *Appl. Math. Sci*., 5, 2855-2861, 2011.

[19] Nofel, T. A, Sayed, M., Hamad, Y. S. and Elagan, S. K., "The improved (G'/G)-expansion method for solving the fifth-order KdV equation," *Annals of Fuzzy Math. Informatics*, 3, 9-17, 2011.

[20] Naher, H., Abdullah, F. A. and Akbar, M. A., "New traveling wave solutions of the higher dimensional nonlinear evolution equation by the improved (G'/G)-expansion method," *World Appl. Sci*. J., 16, 11-21, 2012.

[21] Md. Nur Alam and M. Ali Akbar Traveling Wave Solutions of Nonlinear Evolution Equations Via the New Generalized (G'/G)-Expansion Method 1 (4) (2013): 129-136, DOI: 10.13189/ujcmj.2013.010403.

[22] Abdollah Borhanifar and Reza Abazari, General Solution of Two Generalized Form of Burgers Equation by Using the (G'/G)-Expansion Method Applied Mathematics, 3 (2012), 158-168 http://dx.doi.org/10.4236/am.2012.32025.

[23] Abaker A. Hassaballa et al, Applications of the Improved G_/G Expansion Method for Solve Burgers- Fisher Equation. Journal of Computational and Theoretical Nanoscience Vol. 14, 4664–4668, 2017.

[24] Md. Nur ALAM and M. Ali AKBAR, A New (G'/G)-Expansion Method and Its Application to the Burgers Equation. Applied Mathematics, Walailak J Sci & Tech 2014; 11 (8) http://wjst.wu.ac.th.