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Analysis of a Generalized Formulation of MHD Isothermal Flow over Exponentially Stretching Sheet Under Variable Magnetic Effect

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Abstract: The paper has presented and discussed a single generalized algebraic formulation for magneto-hydrodynamic (MHD) flow over an isothermal exponentially stretching sheet under an exponential magnetic field over a range of a magnetic parameter (M), $0 \leq M \leq 1.0$ and has analyzed relative weights of different terms in the governing equation. Solution methodology is based on minimization of the residual of the governing equation and results are in perfect agreement with other previously published works. Wall shear stress has been formulated as single algebraic equation of M. Inside flow region, shear stress is maximum at the wall and suffers an exponential decrease in vicinity of sheet at similarity variable (η), $\eta \leq 4.0$., where 1st and 3rd terms in the governing equation are the most dominant terms. Within the vicinity of the sheet, the velocity has suffered an exponential decrease that became steeper with the increase of M, signifying a retardation effect of the magnetic field. Beyond $\eta = 4.0$ the flow region is almost stagnant. The analysis shows that high nonlinearity of the governing equation has led to an oscillatory nature especially in the vicinity of the sheet, which becomes more damped at higher values of M. In the range $0 \leq \eta \leq 0.25$, the 2nd nonlinear term in the equation can be neglected, while in the range $0.25 \leq \eta \leq 0.75$, the 4th term can be neglected. In the range, $0.75 \leq \eta \leq 1.0$ both the 3rd and 4th terms of the equation can be neglected. Although neglecting any term of the governing equation will be at the sacrifice of the accuracy of the solution, yet the 2nd term, which is nonlinear, can be totally deleted from the equation at a sacrifice of about 10% of the accuracy of the solution.

Keywords: MHD Boundary Layer Flow, Stretching Sheet, Magnetic Field, Shear Stress

1. Introduction

The boundary layer flow of viscous fluids over flat stretching surfaces have been previously investigated for its important applications in both technological processes (such as: hot rolling, wire drawing, metal spinning, metal and polymer extrusion, manufacturing of plastic films and glass-fibers, … etc.) and processes encountering flow of electrically conducting fluids (such as: magneto-hydrodynamic (MHD) flows, fusing of metals in electrical furnaces under magnetic fields, cooling of walls inside nuclear reactor containment vessels,... etc.). Sakiadis [1], in his pioneer study of axisymmetric two-dimensional (2-D) boundary layer flow over a stretched surface moving with a constant velocity, had developed the boundary layer equations. Extension of his study by Erickson et al. [2] had included effects of the addition of suction and injection. Crane [3] had obtained the analytical solution for a boundary layer flow of an incompressible viscous fluid over a stretching sheet. Further extensions had considered the effects of porous surfaces, MHD fluids, slip effects, … etc. Both Magyari and Keller [4] and Elbashbeshy [5] had examined the flow and heat transfer characteristics over an exponentially stretching permeable surface. Mukhopadhyay [6] has examined slip effects on MHD boundary layer flow over an exponentially stretching sheet. In an attempt to control momentum, Kumaran et al. [7] had shown that the magnetic field had made the streamlines steeper resulting in thinner velocity boundary layer.

Following these studies, several extensional studies had investigated the problem of boundary layer flow of viscous fluids over flat stretching surfaces and obtained closed form solutions (see [8–10]). A detailed literature survey for the flow past a stretching sheet can be seen in a paper by Liao [11]. Mabood et al. [12] have conducted and discussed an analytical Homotopy analytical method (HAM) solution of MHD boundary layer flow over an exponentially stretching sheet in the presence of radiation based on suitable transformations techniques previously used by others (e.g.: Liao, [13], Abbasbandy, [14], Sajid and Hayat, [15], Rashidi et al., [16] and Rashidi et al., [17]). In those studies, the convergence of the series solution and the effects of controlling parameters on MHD flow and heat transfer characteristics have been presented and discussed. Tamizharasi and Kumaran [18] dealt with the pressure in a steady 2-D/axisymmetric MHD/Brinkman flow of an incompressible viscous electrically conducting fluid over a flat stretching sheet. They found that the pressure distribution for the MHD case was finite, while this distribution was totally different in the outer boundary layer for the 2-D and the axisymmetric cases. Kumaran and Tamizharasi et al. [19] developed an approximate analytical solution of the axisymmetric flow of an incompressible viscous electrically conducting fluid over a flat stretching sheet. In their paper, the pressure distribution of the MHD and the porous medium cases are plotted and compared. The MHD flow past a stretching surface of a viscoelastic second grade fluid is studied by Sahoo [20] subject to various physical conditions. The magneto-hydrodynamic flow over a stretching sheet was further studied by Kumaran et al. [21]. The flow along a stretching permeable surface in Darcy–Brinkman porous medium has been investigated by Pantokratoras [22].

Sajid et al. [23] studied the case where the fluid flow was induced by the motion of the surface therefore, the flow behavior was analyzed under the influence of both the motion of the solid surface and buoyancy induced by heating and cooling of the stretching sheet. Recently, Hsiao [24] numerically studied the heat transfer, mass transfer, and mixed convection for MHD flow of a viscoelastic fluid past a continuously moving surface with Ohmic dissipation. Recently, Abbas et al. [25] investigated the mixed convection in the stagnation-point flow of a Maxwell fluid toward a vertical stretching sheet and discussed the results both analytically using the HAM method and numerically using finite difference method. Sajid et al. [26] studied the steady mixed convection stagnation point flow of an incompressible Oldroyd-B fluid over the stretching sheet in the presence of a constant applied magnetic field under linear temperature variation of the surface. By the use of similarity transformations, they used the finite difference scheme to numerically solve the resulting coupled non-linear differential equations for velocity, temperature, skin friction coefficient and rate of heat transfer through the wall.

The case of the unsteady boundary layer flow of viscous fluids over flat stretching surfaces has also been under numerous investigations. Tiegang et al. [27] had proposed and studied a new family of unsteady boundary layers over a constant speed stretching flat surface from a slot moving at a certain speed. They showed that, under specific conditions, the solutions only have existed for a certain range of the slot moving parameter and have reduced to the unsteady Rayleigh problem and the steady Sakiadis stretching sheet problem. Kumaran et al. [28] studied the MHD flow past a stretching permeable sheet and obtained an exact solution for the boundary layer flow of an electrically conducting fluid past a quadratically stretching, and linearly permeable sheet. They graphically showed and discussed the effects of magnetic, suction/injection and linear/nonlinear stretching parameters on the stream function and the skin friction. Gowdara and Bijjanal [29] performed an investigation for the structure of an unsteady boundary layer flow and heat transfer of a dusty fluid over an exponentially stretching sheet subject to suction under the effects of thermal radiation, viscous dissipation, and internal heat generation/absorption. They numerically solved the resulting nonlinear ordinary differential equations by Runge–Kutta–Fehlberg method and presented the effects of the pertinent physical and engineering parameters in graphical and tabular forms.

Although the above-mentioned literature studies have extensively handled too many aspects of the problem of boundary layer flow over a stretching sheet under magnetic effects, yet, none of these studies had attempted to consider a generalized formulation relating the flow parameters so that any future usage can benefit without the task of resolving the flow problem. Also, with the existence of 2 nonlinear terms in the governing equation, previous studies showed no attempt to assess the relative importance of the composing terms of the controlling differential equation that can certainly help in any linearization process to be proposed for the sake of simpler solutions.

The present paper investigates the MHD flow over an isothermal exponentially stretching sheet under an exponential magnetic field to fulfill the following two main objectives: (i) to present and discuss a generalized algebraic formulation for the flow parameters as a function of the controlling parameters; and (ii) present and discuss the relative weights of the different terms involved in the nonlinear initial-boundary value governing differential equation controlling the flow process.

2. Problem Formulation and Analysis

Consider a quiescent region of incompressible, viscous, electrically conducting fluid set in a steady, 2-D flow by a horizontal sheet stretching horizontally with a velocity $U_w(x)$ and is subjected to a variable magnetic field $B(x)$ applied normally to the sheet under no external electric field. The induced magnetic field is assumed negligible, which can be justified for MHD flow at small magnetic Reynolds number (e.g.: Ishak [30]). The electric field due to the polarization of charges is assumed negligible. Under boundary layer approximations, the flow is governed by the following continuity and momentum equations:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (1)$$

$$u\,\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} - \frac{\sigma B(x)^2}{\rho}u \quad (2)$$

where x and y are the horizontal and vertical directions, u and v are the x and y velocity components respectively, ν is the kinematic viscosity, ρ is the fluid density and σ is the electrical conductivity. These fluid properties are assumed constant. The hydrodynamic boundary conditions are: (e.g.: Ishak [30]):

$$u = U_w(x), v = 0\ at\ y = 0 \quad (3)$$

$$u \to 0\ as\ y \to \infty \quad (4)$$

The velocity of the stretching sheet $U_w(x)$ is assumed to vary exponentially in the form:

$$U_w(x) = U_0 e^{\frac{x}{L}} \quad (5)$$

where U_0 is a reference velocity and L is a characteristic length. It is also assumed that the magnetic field $B(x)$ varies exponentially by the relation:

$$B(x) = B_0 e^{\frac{x}{2L}} \quad (6)$$

where B_0 is a constant magnetic field. This magnetic field is represented in dimensionless form by a magnetic parameter M given as:

$$M = \frac{2\sigma {B_0}^2 L}{\rho U_0} \quad (7)$$

The continuity Eq. (1) is satisfied by introducing a stream function ψ such that:

$$u = \frac{\partial \psi}{\partial y} \text{ and } v = -\frac{\partial \psi}{\partial x} \quad (8)$$

To put the momentum and boundary conditions in dimensionless form, the following transformations are introduced (e.g.: [15]):

$$\eta = y\sqrt{\frac{U_0}{2\nu L}}e^{\frac{x}{2L}} \quad (9)$$

$$u = U_0 e^{\frac{x}{L}} f'(\eta) \text{ and } v = -\sqrt{\frac{\nu U_0}{2L}}e^{\frac{x}{2L}}[f(\eta) + \eta f'(\eta)] \quad (10)$$

where η is the similarity variable, $f(\eta)$ is the dimensionless stream function and the prime denotes differentiation with respect to η. Using the above relations and the magnetic parameter, the momentum equation, and the boundary conditions reduce to:

$$f''' + ff'' - 2f'^2 - Mf' = 0 \quad (11)$$

$$f(0) = 0, f'(0) = 1, f'(\infty) = 0 \quad (12)$$

Although the above differential equation is an initial-boundary value ordinary differential equation, yet it is highly nonlinear due its 2nd and 3rd terms. Numerous literature researches reported the use of the HAM method to reach an analytical stable solution (e.g.: [12, 14 and 17]). They also reported that reaching a convergent solution has required higher order iterations of this method. Although usage of Galerkin Weighted Residual Method (WRM) and the Least Squares Weighted Residual method were successful in solving channel flow of magnetic fluid through porous sheet (e.g.: [31] and [32]), yet authors' numerous attempts to solve present flow problem using these two methods, with even higher orders, have resulted in unstable oscillating solutions, which are far from being realistic. Also, attempts to solve this equation by iteration in the form: ${f_{n+1}}''' = -f_n {f_n}'' + 2{f_n'}^2 + M{f_n}'$ has failed for the same reason of the oscillatory nature of the results. Present solution has used the direct minimization of the differential equation, which can be summarized below.

The exact solution of the differential equation is assumed to be of the form:

$$f(\eta) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} a_{ij}\,\eta^i e^{-j\eta} \quad (13)$$

Which, mathematically speaking, is a complete infinite set of independent functions that ultimately converge to the exact solution. The values of the coefficients a_{ij} will be specific to the value of M used in the solution. The application of the initial and boundary conditions of Eq. (12) to this assumed solution results in 3 relations that can be used to cancel two of the unknown coefficients in the above equation. The above equation when substituted in Eq. (12)) will result in equality, while the use of a truncated form as:

$$f_{trun}(\eta) = \sum_{i=0}^{imax}\sum_{j=0}^{jmax} a_{ij}\,\eta^i e^{-j\eta} \quad (14)$$

will result in a residual error as:

$$Resid = f'''_{trun}(\eta) + f_{trun}(\eta) * f''_{trun}(\eta) \\ -2{f'}^2_{trun}(\eta) - Mf'_{trun}(\eta) \quad (15)$$

For any specific assumed value for M, minimization of the absolute value of the above residual will result in values of the unknown coefficients of the assumed solution that are valid only for this specific used value of M. That solution will be acceptable if (i) it satisfies the initial and boundary conditions; (ii) it gives a minimized absolute value for the residual in Eq. (15) within the required error tolerance and (iii) it agrees well with any other published solutions of this equation under similar conditions. Below is a presentation and discussion of using the above methodology to solve this flow problem.

With several presumed values of M in the range $0.0 \leq M \leq 1.0$, present trials utilizing only 2 terms (i.e.: $imax = 2$ and $jmax = 2$) as a solution to Eq. (11), have produced very rough acceptable results that fairly agree in the general trends with other recent published data with same values of M (e.g.: Mabood et al. [12]), but poorly agree with their graphically presented numerical values of their solution. However, the use of series terms having $imax = 2$ and $jmax = 3$ gives fairly well acceptable results in both the general trends and the numerical values as compared with other published results [12]. Although the use of more series terms beyond the above-mentioned will only result in a smaller residual value of the governing differential equation; Eq. (11), which is good, yet it complicates the resulting solution, as well as the resulting enhancements of the solution will be negligible. Applying the initial and boundary conditions given in Eq. (12), to the proposed solution gives:

$$f_{trun}(\eta) = a_{00} + a_{01}\mathrm{e}^{-\eta} + a_{02}\mathrm{e}^{-2\eta} - (a_{00} + a_{01} + a_{02})\mathrm{e}^{-3\eta} + a_{11}\eta\mathrm{e}^{-\eta} + a_{12}\eta\mathrm{e}^{-2\eta} + (1 - 3a_{00} - 2a_{01} - a_{02} - a_{11} - a_{12})\eta\mathrm{e}^{-3\eta} + a_{21}\eta^2\mathrm{e}^{-\eta} + a_{22}\eta^2\mathrm{e}^{-2\eta} + a_{23}\eta^2\mathrm{e}^{-3\eta} \quad (16)$$

Values of the above 8 unknown coefficients; (namely: $a_{00}, a_{01}, a_{02}, a_{11}, a_{12}, a_{21}, a_{22}$ and a_{23}) depend on value of M.

As mentioned before, all previously published results of this flow problem show specific values of solutions in graphical form or tabulated form at specific values of M and none of those results have shown any generalized formulation of the solution that can cover certain ranges of variation of M. As discussed below, present work gives generalized formulation of the solution that covers the most practically important range of M (i.e.: $0.0 \le M \le 1.0$). Generalized relationship relating $f_{trun}(\eta)$ to the value of M is assumed as a 3rd order polynomial in M in the form:

$$f_{trun}(\eta) = \mathrm{a000} + \mathrm{a001}\,M + \mathrm{a002}\,M^2 + \mathrm{a003}\,M^3 + e^{-\eta}(\mathrm{a010} + \mathrm{a011}\,M + \mathrm{a012}\,M^2 + \mathrm{a013}\,M^3) + e^{-2\eta}(\mathrm{a020} + \mathrm{a021}\,M + \mathrm{a022}\,M^2 + \mathrm{a023}\,M^3) - e^{-3\eta}(\mathrm{a000} + \mathrm{a010} + \mathrm{a020} + \mathrm{a001}\,M + \mathrm{a011}\,M + \mathrm{a021}\,M + \mathrm{a002}\,M^2 + \mathrm{a012}\,M^2 + \mathrm{a022}\,M^2 + \mathrm{a003}\,M^3 + \mathrm{a013}\,M^3 + \mathrm{a023}\,M^3) + e^{-\eta}(\mathrm{a110} + \mathrm{a111}\,M + \mathrm{a112}\,M^2 + \mathrm{a113}\,M^3)\eta + e^{-2\eta}(\mathrm{a120} + \mathrm{a121}\,M + \mathrm{a122}\,M^2 + \mathrm{a123}\,M^3)\eta + e^{-3\eta}\big(1 + \mathrm{a010} - \mathrm{a110} - \mathrm{a120} + \mathrm{a011}\,M - \mathrm{a111}\,M - \mathrm{a121}\,M + \mathrm{a012}\,M^2 - \mathrm{a112}\,M^2 - \mathrm{a122}\,M^2 + \mathrm{a013}\,M^3 - \mathrm{a113}\,M^3 - \mathrm{a123}\,M^3 + 2(\mathrm{a020} + \mathrm{a021}\,M + \mathrm{a022}\,M^2 + \mathrm{a023}\,M^3) - 3(\mathrm{a000} + \mathrm{a010} + \mathrm{a020} + \mathrm{a001}\,M + \mathrm{a011}\,M + \mathrm{a021}\,M + \mathrm{a002}\,M^2 + \mathrm{a012}\,M^2 + \mathrm{a022}\,M^2 + \mathrm{a003}\,M^3 + \mathrm{a013}\,M^3 + \mathrm{a023}\,M^3)\big)\eta + e^{-\eta}(\mathrm{a210} + \mathrm{a211}\,M + \mathrm{a212}\,M^2 + \mathrm{a213}\,M^3)\eta^2 + e^{-2\eta}(\mathrm{a220} + \mathrm{a221}\,M + \mathrm{a222}\,M^2 + \mathrm{a223}\,M^3)\eta^2 + e^{-3\eta}(\mathrm{a230} + \mathrm{a231}\,M + \mathrm{a232}\,M^2 + \mathrm{a233}\,M^3)\eta^2 \quad (17)$$

The application of the above methodology to specific values of M in the range $0.0 \le M \le 1.0$ gives numerical values for the unknown coefficients shown in Eq. (16). To fulfill the first objective of this study, the resulting values of these coefficients, are fitted to each corresponding value of M, for the sake of evaluation of the values of the constants shown in Eq. (17), and hence to have the generalized flow equation. To fulfill the second objective of the present study, the resulted fitted equations are to be used to assess the relative weight of each term in the differential Eq. (15). This can be of great help for any future proposed simplification or linearization of this equation.

3. Results and Discussion

Results of applying the above methodology using numerous M values in the range $0.0 \le M \le 1.0$ to minimize the absolute value of the residual, given by Eq. (15), summed over 2000 complete calculated sets, using $\xi\eta = 0.1\,\eta$ and $\xi M = 1 + 4M$, is as follows:

$$Minimum \sum_{\xi M=1}^{5} \sum_{\xi\eta=0}^{400} \mathrm{Abs}[Resid] = 1.60768049275 \quad (18)$$

Values of the coefficients in Eq. (16) are given in Table (1).

***Table 1.** Coefficients of Eq. (16) as functions in M for the case* $0.0 \le M \le 1.0$.

	M^0	M^1	M^2	M^3
a_{00}	0.79125190130	-0.08674189435	0.00406712781	-0.05335645174
a_{01}	-0.42752690460	0.33666056921	-0.09802532963	0.03680960142
a_{02}	-0.17038833700	-0.18995396189	-0.17609631063	0.24492386583
a_{11}	-0.23401574285	0.06531846478	0.32726038076	0.06630493486
a_{12}	0.19675141374	-0.33074000076	0.02005686463	-0.08753499807
a_{21}	0.17845164512	-0.15745452953	-0.20994512872	0.13071959156
a_{22}	-0.04574033513	-0.04309754827	-0.10455684738	-0.18175787932
a_{23}	-0.12503863343	0.03476059461	-0.09411570718	0.05584199724

The dimensionless generalized relationship of the wall shear stress (i.e.: skin friction) $f''_{trun}(\eta = 0)$ is given by:

$$f''_{trun}(0) = -1.28644363445 - 0.35577784303M + 0.00032640690M^2 + 0.01171136818M^3 \quad (19)$$

3.1. Comparison with Recently Published Results

Plotting of present generalized relationship of the shear stress given in Eq. (19) and that of Mabood et al. [12], as directly taken from their study, are shown in Fig. 1(a), (b). The figure shows a perfect agreement both in trend as well as in values between these two results over the shown range of M. This signifies that, although present methodology of solving the flow problem is both simple and covers a range of M values, yet it agrees well with results of the more elaborated analytical HAM method (e.g.: [12]) that calculates flow function case be case.

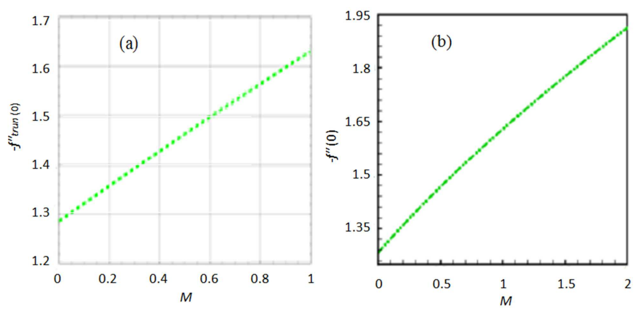

***Fig. 1.** Wall shear stress as dependent on magnetic parameter. (a) from Eq. (19) of present study and (b) as Fig. (9) in Ref. [12].*

Present values of dimensionless flow function $f_{trun}(\eta)$ and dimensionless velocity $f'_{trun}(\eta)$ are shown in Fig. (2(a)) for values of $= 0.0, 0.25, 0.5, 0.75$ *and* 1.0. The behavior of these groups of curves agrees well with previously published results shown in Fig. (2(b)) for the flow velocity. In both studies, the figure shows that beyond the value of $\eta = 4.0$ the flow region is almost stagnant as a result of the decreased effects of the stretching sheet on the motion of the fluid. Within the vicinity of the sheet, in the range $0.0 \le \eta \le 4.0$, the velocity suffers an exponentially sharp decrease at all values of M, which is a true reflection of the solution in Eq. (17). Both parts of the figure also show that higher values of M result in the steeper decrease of the velocity distribution, a condition which is in agreement with work of Kumaran et al. [7]. This signifies the retardation effect of the magnetic field on the flow even in the vicinity of the stretching sheet.

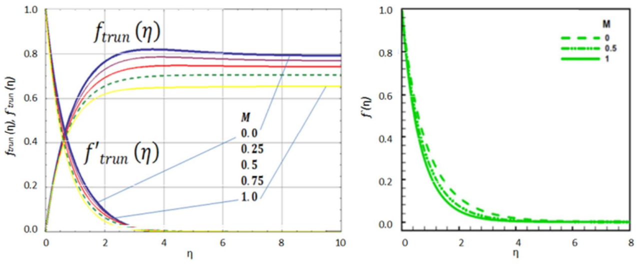

Fig. 2. Dimensionless flow function $f_{trun}(\eta)$ and velocity $f'_{trun}(\eta)$. (a) present study at M = 0.0, 0.25, 0.5, 0.75, 1.0, (b) as Fig. (6) in Ref. [12] at M = 0.0, 0.5, 1.0.

3.2. Errors Analysis

The errors encountered in the present study for the governing differential Eq. (15) are shown in Fig. (3) for values of $M =$ 0.0, 0.25, 0.5, 0.75 and 1.0. The high nonlinearity of the governing Eq. (15) has led to the oscillatory nature of the shown error, a condition that has been shown in previous studies of this kind of flow. Also, the shown errors for given values of M exhibit almost same value for $0.0 \le \eta \le 0.5$, a condition that can be explained as the strong effect of the stretching sheet near its vicinity. Due to the weak effect of the sheet in the region, $\eta \ge 8.0$, this error vanishes at all values of M, signifying that the nonlinearity effects in the governing differential are insignificant and the fluid is almost stagnant in that region. An important conclusion can be withdrawn from this figure that increasing the magnetic field, (i.e.: at higher values of M), results in more damping of the oscillatory of the behavior of the governing equation. One might argue that the present solution method might be the cause of this oscillation. In fact, this oscillatory nature has been shown in other previously published works that had utilized more elaborate other solution methods (e.g. HAM by [12]). Also, present authors' attempts to use discontinuous Galerkin method or Least Square WRM to solve this specific flow problem have failed due to the very rough oscillations of both the solution and the residual errors.

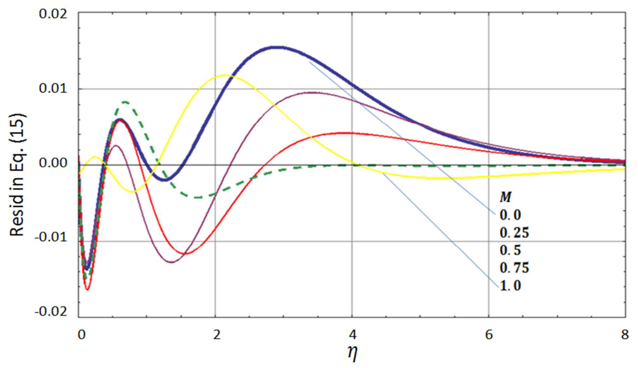

Fig. 3. Errors encountered in present study for M = 0.0, 0.25, 0.5, 0.75 and 1.0.

3.3. Shear Stress Inside Flow Region

Figure (4) shows the shear stress inside the flow region for values of $M = 0.0, 0.25, 0.5, 0.75$ and 1.0. At the wall, where $\eta = 0$, the shear stress is included as an inset in the figure, which is the same as shown in Fig. (1) above at respective values of M. The flow region can be considered to be composed of two regions; in the vicinity to the sheet where $\eta \le 4.0$; and away from the sheet. At all values of M, the shear stress has its maximum value on the wall, then suffers an almost same exponential decrease inside the first region, and almost vanishes in the second region.

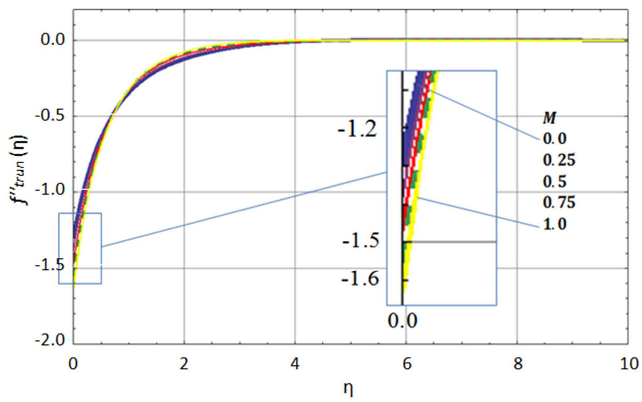

Fig. 4. Shear stress inside flow region.

3.4. Relative Weights of the 4 Terms of Eq. (15).

Figure (5) shows the relative terms of the governing differential Eq. (15) at 5 values of M. All 4 terms at all M values exhibit one common behavior; which is the reduction to a null value away from the stretching sheet beyond about $\eta \ge 4.0$, which signifies that the fluid is stagnant in that region. Within the vicinity of the sheet, the most dominant terms are the first and third terms, although they almost cancel each other. The effect of M, given in the 4th term, shows a noticeably increased damping in the flow with the increase of M. The 2nd term is affected little away from the sheet, i.e. in the range $0.5 \le \eta \le 4.0$. This figure also shows that the 2nd term, which is a nonlinear one, can be totally deleted from the governing Eq.(15) for a sacrifice of about 10% of the accuracy of the solution of that equation.

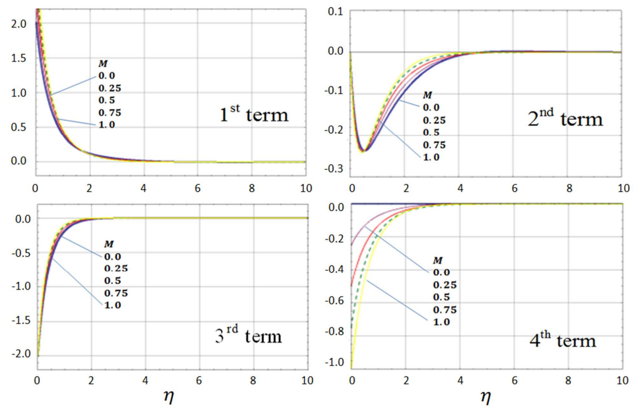

Fig. 5. Relative weights of the 4 terms of governing differential Eq. (15) at values of M = 0.0, 0.25, 0.5, 0.75 and 1.0.

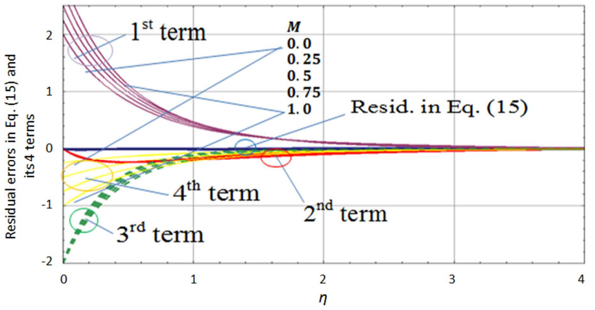

Fig. 6. Error values of governing differential Eq. (15) and errors in its terms at values of M = 0.0, 0.25, 0.5, 0.75 and 1.0.

Figure (6) shows the relative errors of the 4 terms of the governing Eq. (15) with regards to the residual error in the whole equation at 5 different values of M. With the scales shown in the figure, the residual in the whole equation for all values of M can be considered an almost zero (shown as a horizontal line with a value = 0), which proves the good credibility of the solution method in present study. However, the nonlinearity of the equation is manifested by the different behaviors of the four terms of the equation at different values of M and η. As mentioned before, away from the sheet, (i.e. where range $\eta \geq 4.0$), the flow is almost stagnant, which is exhibited by the null values of all errors of the 4 terms of the equation. Near vicinity of the sheet, the 2nd nonlinear term in the equation (i.e.: $f_{trun}(\eta) * f''_{trun}(\eta)$) can be neglected in the range $0 \leq \eta \leq 0.25$, while in the range $0.25 \leq \eta \leq 0.75$, the 4th term (i.e.: $-Mf'_{trun}(\eta)$, which include M, can be neglected. In the range, $0.75 \leq \eta \leq 1.0$ both the 3rd and 4th terms of the equation can be neglected. It is worthy to mention that neglecting any term of the governing Eq. (15) will be at the sacrifice of the accuracy of the solution of the whole equation.

4. Conclusions

The paper has presented and discussed a generalized algebraic formulation for MHD flow over an isothermal exponentially stretching sheet under an exponential magnetic field in the range of a magnetic parameter (M), $0.0 \leq M \leq 1.0$ and has analyzed relative weights of different terms in the governing flow equation. Solution methodology was mainly based on the minimization of the residual of the differential equation. Dimensionless wall shear stress has been algebraically formulated as dependent on M. The shear stress, the flow function and velocity distribution are in perfect agreement with other previously published works. Beyond the value of similarity variable (η), $\eta = 4.0$ the flow region is almost stagnant due to the decreased effects of both the stretching sheet and the magnetic field on the fluid. Within the vicinity of the sheet, the velocity has suffered an exponential decrease which became steeper with the increase of M, signifying retardation effect of the magnetic field.

The analysis shows that high nonlinearity of the governing equation has led to an oscillatory nature especially in vicinity of the sheet, which becomes more damped at higher values of M. At all values of M, the shear stress has its greatest value on the wall, then suffers an exponential decrease in vicinity of the sheet at $\eta \leq 4.0$., where the most dominant terms in the governing equation are the first and third terms. In the range $0 \leq \eta \leq 0.25$, the 2nd nonlinear term in the equation (i.e.: $f_{trun}(\eta) * f''_{trun}(\eta)$) can be neglected, while in the range $0.25 \leq \eta \leq 0.75$, the 4th term (i.e.: $-Mf'_{trun}(\eta)$, can be neglected. In the range, $0.75 \leq \eta \leq 1.0$ both the 3rd and 4th terms of the equation can be neglected. Although it is worthy to mention that neglecting any term of the governing equation will be at the sacrifice of the accuracy of the solution, yet the 2nd term, which is a nonlinear one, can be totally deleted from the equation for a sacrifice of about 10% of the accuracy of the solution.

Nomenclature

a	coefficient
B_0	constant magnetic field
$B(x)$	variable magnetic field
2-D	two dimensional
$f(\eta)$	dimensionless stream function
HAM	Homotopy analytical method
i, j, n	recursive indices
L	characteristic length
M	magnetic parameter
MHD	Magneto-Hydrodynamic
$Resid$	residual error
$trun$	truncated
x, y	horizontal and vertical directions
u, v, x, y	velocity components
U_0	reference velocity
$U_w(x)$	sheet stretching velocity
WRM	Weighted residual method
Greek letters	
η	similarity variable
ν	fluid kinematic viscosity
ρ	fluid density
σ	fluid electrical conductivity
ψ	stream function
$\xi\eta, \xi M$	recursive indices

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