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Modeling and Stability Analysis of a Three Species Ecosystem with the Third Species Response to the First Species in Sigmoid Functional Response Form

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Abstract: In this paper, a three species eco system, involving three pairs is considered modeled to examine the stability. Among the three species, one plays dual roles which are a host and an enemy with Monod response. In the first place model assumptions and formulation was carried out for investigations. The biological feasibility of the system is checked. That is positivity and boundedness of the model is verified. It is shown that biologically valid. The dynamical behavior of the proposed model system was analyzed qualitatively. The dynamical analysis includes the determination of all possible equilibrium points and their stability properties. All the equilibrium states are identified and the local asymptotic stability of some of the equilibrium states is examined by considering the set criteria. It is observed that among the states, the state in which the Prey and its Host species are exist is stable and the state where the Predator/Ammensal species is washed out is asymptotically stable. The global stability of the co-existence of the species was investigated by constructing a suitable Lyapunov function. To support our analytical studies, some numerical simulations was performed susing some mathematical software and the results were forwarded in the last section.

Keywords: Prey, Predator, Ammensal, Commensal, Host, Continuous Time, Stability, Numerical Simulation

1. Introduction

The development of the qualitative analysis of ordinary differential equations is deriving to study many problems in mathematical biology.

Ecosystems are the ones in which their living and nonliving components interact with and depend on each other linking together the exchange of energy, material, information. The structure and the function of the ecosystems are determined by the interplay of both cooperation and competition [21, 22]. Ecosystems are able to regulate themselves to maintain certain stability. Therefore, the stability is one of the most fundamental and essential features of the ecological systems. The study of stability is directed relevant to the existence of every species. The stability is influenced by many factors, such as the structure within the components and the features of the environment. The ecosystems are complex and involve many kinds of interactions among the elements. The inherent interactions are often non-linear and intricate. These systems can be described by a set of nonlinear differential equations. These nonlinear interactions lead to complex dynamics. There have been many investigations on the stability of ecosystems. Most of the works have been focused on the local linear stability analysis. The studies of the stability of ecosystems are significant for uncovering the underlies ecological law of species and populations [34].

The global stability of the ecological systems is still challenging in general. Furthermore, the link between the global characterization of the ecological systems and the dynamics of the elements is still not clear. The past researchers explored the dynamical system with the approach of Lyapunov function which was developed to investigate the global stability. Here, in recent work we would like to suggest a universal and straightforward approach to explore the Lyapunov function and therefore the global stability of the general ecological systems.

In nature, all living species like a suitable environment where it can live freely and reproduce. Ecological species take various techniques for searching foods and for defensive purposes.

Ecology, basically the study of the inter relationship between species and their environment, in such areas as predator-prey and competition interactions, renewable resources management, evolution of pesticide resistant strains ecological and genetically engineered control of pests, multi-species societies, plant-herbivore systems and so on is now an enormous field. It is the scientific study of the interactions between organisms and their environment [23-25].

In the ecosystems, the relationship between species can be grouped into two categories: the negative antagonism interaction (−) and the positive mutualism interaction. Predation shows the relationship (+/−) which one species is disfavored, while the other species benefits. Examples in the natural world include sharks and fish, lynx and snowshoe hares, and ladybirds and aphids. Mutualism shows the relationship (+/+) which both species benefit from interactions of the other. Stability and dynamics are crucial for understanding the structure and the function of ecosystems [23, 29-34].

In this paper we are interested in ecological systems in which the interactions were both positive and negative to show the stability of the ecosystem.

A brief description of a commensalism interaction was given by different scholars [26-29, 31-32, 34].

Ammensalism is the ecological interaction in which an individual species harms another without obtaining benefit. This type of symbiotic relationship is common, but not considered an important process structuring communities because they are “accidental” and do not benefit the species doing the harm. It is a 0- relationship. For instance, algal blooms can lead to the death of many species of fish and other animals, however the algae do not benefit from the deaths of these individuals.

These different theoretical studies was manifested or visualized by applying mathematical language and defined by varies researcher [4-5, 8-9, 11-17].

Dynamics of non-linear systems that occur in ecological systems has attracted the attention of mathematicians since the days of Lotka [1] and Volterra [2]. Over the years, this model has attracted attentions for exploring the dynamical process of the ecology. Non-linear dynamic models exhibit a wide range of behaviors. The Lotka-Volterra Prey-Predator model involves two equations, one which describes how the prey population changes and the second which describes how the predator population changes would be defined by the differential equations as follow;

$$\frac{dH}{dt} = aH(t) - bH(t)P(t$$

$$\frac{dP}{dt} = ebH(t)P(t) - cP(t) \quad (1)$$

Where a, b, c and e are all positive constants, with $H(t)$ and $P(t)$ representing the scaled population of prey and predator, respectively, and t is measured in years.

Moreover; the other models involving Commensal-host and ammensalism model were described as shown in equation (2) and (3) respectively.

Commensal-host model:

$$\frac{dH}{dt} = rH(t)\left[1 - \frac{H(t)}{k}\right]$$

$$\frac{dC}{dt} = aC(t)\left[1 - \frac{C(t)}{k\prime}\right] + eH(t)C(t) \quad (2)$$

Where $r, k, k^{'}, a$ and e positive constants with are $H(t)\ and\ C(t)$ denotes the population of host and Commensal respectively.

Ammensalism model:

$$\frac{dN}{dt} = rN(t)\left[1 - \frac{N(t)}{k}\right]$$

$$\frac{dA}{dt} = aA(t)\left[1 - \frac{A(t)}{k\prime}\right] - gN(t)A(t) \quad (3)$$

Where $r, k, k', a\ and\ g$ are positive constants with $N(t)$ and $A(t)$ are density of populations where $A(t)$ is Ammensal.

Inspired by these model, several researchers made significant contributions in this area by considering various special types of interactions between the species. This has been the motivation for others in bringing a third species into the system thus forming a three species ecological system.

In the present paper, the three species Ecosystem with time as continuous unit is considered. The equilibrium states are identified and the asymptotic stability of the equilibrium states is examined. A few of them are presented here.

Now, the present investigation is a study of a continuous model of “a symbiotic interaction and predation” between three species.

2. Assumptions and Models Equations

Mathematical modeling and computer simulation provide an effective tool in the study of contemporary population ecology [18, 19]. In population dynamics, the functional response of predator to prey density refers to the change in the density of prey attacked per unit time per predator as the prey density changes [20].

Recently, Koya P. Roa and Geremew K [34] worked on three species system by considering interactions like Prey-Predator, Commensal - Host, and between the three species, which motivated the present authors to consider a three species Ecosystem with species S1, S2 and S3 simultaneously having the interactions of Prey-predation, commensalism and ammensalism, with continuous time. Here, S1 and S2 form a Prey-Predator pair. That is, S2 depends on S1 for its survival. S1 and S3 form a Commensal – Host pair. That is, S3 acts as host to S1 without itself being affected. Moreover, the response between S3 and S1 is sigmoid functional response not a linear. And S2 and S3 form an Ammensal – Enemy pair. That is, S3 inhibits S2 without itself being affected, as shown illustrated in figure 1 below.

In this paper, we describe the three species ecosystem model. To develop this model, the assumptions have been made as mentioned

The three model equations for $dN1/dt, dN2/dt$ and $dN3/dt$ are constructed using several components, both variables and parameters, and each of which represents specific biological assumptions.

The simple schematic interactions among the model variable is shown in Figure 1.

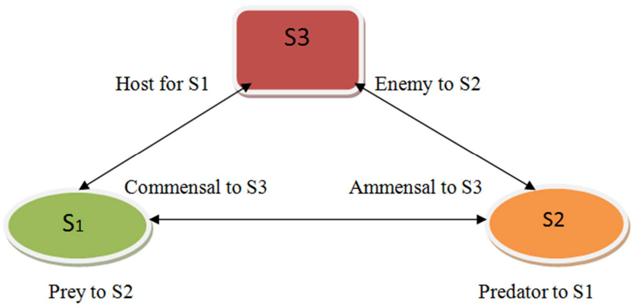

Figure 1. *A three species ecosystems.*

2.2. The Model Equations

Based on the above initial models given in system (1) - (3) and the assumptions, the interactions of the three species stated by the model equations as follows;

$$\frac{dN_1}{dt} = a_1N_1 - a_{11}{N^2}_1 + \left[\frac{a}{1+ahN_1}\right] N_1N_3 - a_{12}N_1N_2 \quad (4)$$

$$\frac{dN_2}{dt} = a_2N_2 - a_{22}{N^2}_2 + a_{21}N_1N_2 - a_{23}N_2N_3 \quad (5)$$

$$\frac{dN_3}{dt} = a_3N_3 - a_{33}{N^2}_3 \quad (6)$$

with initial conditions

$$N_1(0) = N_{01} > 0$$

$$N_2(0) = N_{02} > 0$$

$$N_3(0) = N_{03} > 0$$

Notations and descriptions of parameters

Notations and descriptions of the state variables and parameter are listed in the table 1 below.

Table 1. *Notation and description of the model parameters.*

Notation	Descriptions
a_i	The Natural growth rate of Si, i=1, 2, 3; the difference of birth and death rate.
a_{ii}	Self inhibition coefficient of Si, i=1, 2, 3. (The rate of decrease of Ni due to insufficient natural resources of Si)
a_{12}	The rate of decrease of S1 due to inhibition by S2
A	The functional response of the S1 to its host S3; A = $\frac{a}{1+ahN_1}$
a_{21}	The rate of increase of the S2 due to its attacks on S1
a_{23}	The rate of decrease of the S2 due to the harm caused by its enemy S3
a_{12}/a_{11}	Coefficient of prey/commensal inhibition of the predator
$A/a11$	Coefficient of commensalism
$a21/a22$	Coefficient of predator consumption of the prey
$\frac{a23}{a22}$	Coefficient of Ammensalism

The model variables $Ni\ (t)$ the density of population Si, i=1, 2, 3 at any instant of time t subject to the non-negative initial conditions $Ni(0) = N_0 > 0$.

Moreover, the notation A in the model indicates the Functional response with Holling type II response, i.e.,$A = \frac{a}{1+ahN_1}$, where a is helping rate, h-handling rate.

3. Dynamical Properties of the Model System

Since the state variables N_1, N_2 and N_3 represent population sizes, positivity implies that the population sizes never become negative. The boundedness of the system is interpreted as a natural restriction to the growth of populations as consequences of limited resources.

Positivity and boundedness of the solution of the system

In this section, some basic dynamical properties of the system are discussed subjected to positive initial conditions.

Positivity of the solution

Here the positivity of each population size such as $N_1(t)$, $N_2(t)$ and $N_3(t)$ is verified. These system variables must have the positive values in order to be biologically meaningful. The positivity of these biological or system variables is tested and the results are presented in the form of proposition as follows:

Proposition 1 Every solution of system (4-6) together with the positive initial conditions exists in the interval $[0,\infty)$ and they are non-negative. That is, $N_1(t), N_2\ (t)$ and $N_3\ (t\ ) \geq 0)$ for all $t \geq 0$.

Proof

For $t \in [0, T]$, as the system (4) is continuous, then the solution $N_1(t), N_2\ (t)$ and $N_3\ (t$ of the system with given initial conditions exists and unique on $[0,T]$ where $0 < T < +\infty$.

a) Positivity of $N_1(t), N_2\ (t)$ and N_3 (t)

Verifying the positivity of $N_1(t)$: The density of the first population of the system (2.1) is solved analytically and its solution is obtained as:

$$N_1(t) = N_{01}exp \int_0^T \left[a_1 - a_{11}N_1(t) + \frac{aN_3(t)}{1+ahN_1} - a_{12}N_2\right] dt$$

The exponential function is always non-negative and the initial population $N_1(t)$ is assumed to be positive. Therefore, $N_1(t) > 0$ for all $t \geq 0$.

Verifying the positivity of $N_2(t)$: The density of the second population of the system (5) is solved analytically and its solution is obtained as follow:

$$N_2(t) = N_{02}\left[exp \int_0^T (a_2 - a_{22}N_2(u) + a_{21}N_1(u) - a_{23}N_3(u))\right] du$$

The exponential function is always non-negative and the initial population N_{02} is assumed to be positive. Therefore, $N_2(t) > 0$ for all t ≥ 0.

Verifying the positivity of $N_3(t)$: The density of the third population of the system (6) is solved analytically and its solution is obtained as:

$$N_3(t) = N_{03}\left[exp\int(a_3 - a_{33}N_3)\right]dt$$

The exponential function is always non-negative and the initial population N_{03} is assumed to be positive. Therefore, $N_3(t) > 0$ for allI ≥ 0. Hence, all the solutions of the system (4-6) are positive for all t ≥ 0 under the considered positive initial conditions.

b) Boundedness of $N_1(t), N_2\ (t)$ and N_3 (t)

In this section, all the solutions of system (4), (5) and (6) are shown to be bounded. The boundedness of the system is presented in the form of a proposition as follows:

Proposition 2: All solutions of the system (4), (5) and (6) with positive initial conditions are bounded.

Proof: Boundedness of the third population: To show that the population is bounded it is appropriate to start with the third equation from the model system.

Thus,

$$\frac{dN_3}{dt} = a_3N_3 - a_{33}N^2{}_3$$

Using partial fractions and performing of some simple algebraic manipulations reduces the equation to

$$\left[\frac{a_{33}}{1 - a_{33}N_3} + \frac{1}{a_3N_3}\right]N_3 = dt$$

Application of integration reduces it to: $log\left[\frac{a_3N_3}{1-a_{33}N_3}\right] = t + logk$

Here the quantity is an arbitrarily integral constant and must be positive. Applications of anti-logarithm lead to: $\left[\frac{a_3N_3}{1-a_{33}N_3}\right] = ke^t$

Equivalently; $N_3(a_3 + a_{33}ke^t) = ke^t$

$N_3 = \frac{ke^t}{a_3+a_{33}ke^t}$, this implies that $N_3 = \frac{k}{a_3e^{-t}+a_{33}k}$.

Now $e^{-t} \to 0$ as t → ∞. Thus, the system for N_3takes the form as: $N_3 = \frac{1}{a_{33}} \le a_{33}$

Therefore the third population N_3 is bounded above by inverse of its self inhibition coefficient. That is the host population is bounded above by its carrying capacity.

Boundedness of the first population: To show that the first population is bounded it is appropriate to start with the equation (4) from the model system.

$$\frac{dN_1}{dt} = a_1N_1 - a_{11}N^2{}_1 + \frac{aN_1N_3}{1 + ahN_1} - a_{12}N_1N_2$$

It is true that the term $(a_{12}N_1N_2)$ and $(a_{11}N^2{}_1)$ are positive since each member of it is a positive quantity. Thus, without loss of generality

$$\frac{dN_1}{dt} \le a_1N_1 + \frac{aN_1N_3}{1 + ahN_1}$$

$\left(\frac{1}{N_1}\right)\frac{dN_1}{dt} \le a_1 + \frac{a_1N_3}{1+ahN_1} \le a_1 + \frac{aa_{33}}{1+ahN_1} \le \frac{a_{33}}{h}$, since $\frac{1}{1+ahN_1} \le 1$

Thus, the inequality can be re-expressed as;

$$\left(\frac{1}{N_1}\right)\frac{dN_1}{dt} \le a_1 + \frac{a_{33}}{h}$$

On applying integration, it can be obtained that

$$\log N_1 \le \left[a_1 + \frac{a_{33}}{h}\right]t + logm$$

This is equivalent to $N_1 \le me^{\left[a_1+\frac{a_{33}}{h}\right]t}$

Here the quantity m is an arbitrarily integral constant and must be strictly positive due to logarithmic function. Now there arise three cases, namely the exponent may be negative or positive or zero. These cases as analyzed as follows:

i. If $\left[a_1 + \frac{a_{33}}{h}\right] < 0$ then as t → ∞ the exponential term takes a value zero and thus to get $N_1 \le 0$. But this, having negative population, is biologically not feasible. Hence this possibility is not considered.
ii. If $\left[a_1 + \frac{a_{33}}{h}\right] > 0$ then as t → ∞ the exponential term takes a value ∞ and thus to get $N_1 \le \infty$. But this, having ∞ population, is biologically not feasible. It is an unbounded case. Hence this possibility is not considered.
iii. If$\left[a_1 + \frac{a_{33}}{h}\right]$=0 then as t → ∞ the exponential term takes a value 1 and thus reduces to $N_1 \le$ m. Hence this possibility is considered. Thus the population N_1 is bounded above by arbitrarily positive constant m. In other word the death rate more prominent than the other parameters that is why this possibility is happen.

Boundedness of the second population: To show that the population is bounded it is appropriate to start with the equation (5) from the model system.

Thus,

$$\frac{dN_2}{dt} = a_2N_2 - a_{22}N^2{}_2 + a_{21}N_1N_2 - a_{23}N_2N_3$$

$$\frac{dN_2}{dt} = N_2[a_2 - a_{22}N_2 + a_{21}N_1 - a_{23}N_3]$$

It has been already shown that the population sizes of the two populations are bounded. That is, $N_3 \le a_{33}$and $N_1 \le$ m. Here it can be observed that $a_{21}N_1 \le a_{21}m$ and $a_{23}N_3 \le a_{23}a_{33}$.

In view of these observations the equation takes the form as:

$$\frac{dN_2}{dt} \le N_2[a_2 - a_{22}N_2 + a_{21}m - a_{23}a_{33}]$$

$\frac{dN_2}{dt} \le N_2[n - a_{22}N_2]$. Here $n = [a_2 + a_{21}m - a_{23}a_{33}]$.

Applying partial fraction and integrating:

$$ln\left[\frac{N_2}{n - a_{22}N_2}\right] \le nt + lnp$$

Here the quantity p is an arbitrarily integral constant and must be strictly positive due to logarithmic function.

$$\left[\frac{N_2}{n - a_{22}N_2}\right] \le pe^{nt}$$

$$N_2 \le \frac{pne^{nt}}{1 + a_{22}pe^{nt}}$$

$$N_2 \le \frac{pn}{e^{-nt} + a_{22}p}$$

It can be observed that $e^{-nt} \to 0$ as $t \to \infty$ and thus $N_2 \le \frac{n}{a_{22}}$

That is, the second population is bounded above by $N_2 \le \frac{[a_2 + a_{21}m - a_{23}a_{33}]}{a_{22}}$.

Therefore, the solution of the model system (4-6) is bounded.

4. The Steady States

The critical points of the system can be obtained by setting $\frac{Ni(t}{dt} = 0$, i=1, 2, 3. in the model equations. This leads to the following optional relations;

$$N_1 = 0 \quad (7)$$

$$\text{Or, } \left[a_1 - a_{11}N_1 + \frac{a}{1+ahN_1}N_3 - a_{12}N_2\right] = 0 \quad (8)$$

$$N_2 = 0 \quad (9)$$

$$\text{Or, } [a_2 - a_{22}N_2 + a_{21}N_1 - a_{23}N_3] = 0 \quad (10)$$

$$N_3 = 0 \quad (11)$$

$$\text{Or, } [a_3 - a_{33}N_3] = 0 \quad (12)$$

The solutions of these optional relations can be the equilibrium points. There are eight possible combinations of the relations. These combinations and their solutions or equilibrium points are as mentioned below:

Table 2. Possible combinations of the solution of the systems.

Combination of relations	Equilibrium points
(2.4) (2.6) (2.8)	E_0
(2.4) (2.6) (2.9)	E_1
(2.4) (2.7) (2.8)	E_2
(2.5) (2.6) (2.8)	E_3
(2.4) (2.7) (2.9)	E_4
(2.5) (2.6) (2.9)	E_5
(2.5) (2.7) (2.8)	E_6
(2.5) (2.7) (2.9)	E_7
(2.4) (2.5) (2.9)	Not biologically feasible

It can be observed that the first two combinations lead to the same equilibrium point while the last combination leads to biologically infeasible solution. Thus, eight equilibrium points are possible. Now the coordinates of these possible equilibrium points as given below:

(1) Fully washed out state: E_0 (0, 0, 0), i.e., N1=0, N 2=0, N3=0. The extinction of all populations equilibrium (E_0) always exists.

(2) States in which two species are washed out: $E_1\left(0,0,\frac{a_3}{a_{33}}\right), E_2\left(0,\frac{a_2}{a_{22}},0\right), E_3\left(\frac{a_1}{a_{11}},0,0\right)$

(3) States in which one species is washed out: $E_4\left(0,N_2^*,\frac{a_3}{a_{33}}\right), E_5\left(N_1^*,0,\frac{a_3}{a_{33}}\right), E_6(N_1^{**},N_2^{**},0)$

Where $N_1^* = \frac{-k+\sqrt{k^2-4M}}{2}$, $k = \frac{a_1}{a_{22}a_{33}} - \frac{1}{ah}$; $M = \frac{a_1+a_3a}{a_{22}a_{33}ah}$; $N_1^{**} = \frac{a_1a_{22}-a_2a_{12}}{a_{11}a_{22}+a_{12}a_{21}}$;

$N_2^* = \frac{a_2}{a_{22}} - \frac{a_3a_{23}}{a_{22}a_{33}}$ when $a_2a_{33} > a_3a_{23}$; $N_2^{**} = \frac{a_1}{a_{12}} - \frac{a_{11}}{a_{12}}\left[\frac{a_1a_{22}-a_2a_{12}}{a_{11}a_{22}+a_{12}a_{21}}\right]$.

(4) States in which the three species exists: $E_7(N_1^{***},N_2^{***},N_3^{***})$, where

$$N_1^{***} = \frac{-P - \sqrt{P^2 - 4QS}}{2Q};\ Q = -\left[a_{11}ah + \frac{a_{21}a_{12}ah}{a_{22}}\right],$$

$$P = a_1ah + \frac{a_3a_{12}a_{23}ah}{a_{22}a_{33}} - \left[\frac{a_{22}a_{11}+a_{12}a_2ah+a_{21}a_{12}}{a_{22}}\right],$$

$$S = a_1 + \frac{a_3a}{a_{33}} - \frac{a_{12}a_2}{a_{22}} + \frac{a_3a_{23}a_{12}}{a_{22}a_{33}}.$$

$$N_2^{***} = \frac{a_2}{a_{22}} + \frac{a_{21}}{a_{22}}N_1^{***} - \frac{a_3a_{23}}{a_{22}a_{33}};$$

$$N_3^{***} = \frac{a_3}{a_{33}}.$$

5. Stability Analysis of the Equilibrium States

To analyze the stability near the equilibrium points the community matrix called Jacobian matrix is and the conditions for stability of the equilibrium state are determined and stated as follows.

Community matrix

Let $\frac{dN_1}{dt} = h\,(N_1, N_2, N_3)$, $\frac{dN_2}{dt} = f\,(N_1, N_2, N_3)$, and $\frac{dN_3}{dt} = g\,(N_1, N_2, N_3)$.

Where the functions are given as; $h = a_1N_1 - a_{11}{N^2}_1 + \frac{a}{1+ahN_1}N_1N_3 - a_{12}N_1N_2$,

$f = a_2N_2 - a_{22}{N^2}_2 + a_{21}N_1N_2 - a_{23}N_2N_3$ and $g = a_3N_3 - a_{33}{N^2}_3$.

Moreover, the components of the Jacobian matrix are given by;

$$J = \begin{bmatrix} \frac{\partial h}{\partial N_1} & \frac{\partial h}{\partial N_2} & \frac{\partial h}{\partial N_3} \\ \frac{\partial f}{\partial N_1} & \frac{\partial f}{\partial N_2} & \frac{\partial f}{\partial N_3} \\ \frac{\partial g}{\partial N_1} & \frac{\partial g}{\partial N_2} & \frac{\partial g}{\partial N_3} \end{bmatrix}$$

Therefore, the Jacobian matrix takes the components as;

$$J = \begin{bmatrix} a_1 - 2a_{11}N_1 + \frac{aN_3}{(1+ahN_1)^2} - a_{12}N_2 & -a_{12}N_1 & \frac{aN_1}{1+ahN_1} \\ a_{21}N_2 & a_2 - 2a_{22}N_2 + a_{21}N_1 - a_{23}N_3 & -a_{23}N_2 \\ 0 & 0 & a_3 - 2a_{33}N_3 \end{bmatrix} \tag{13}$$

Analysis of the equilibrium points

5.1. Local Stability of the Steady State

In this section, the stability of the model (2.1), (2.2) and (2.3) at the equilibrium point is analyzed. The local stability of the steady state is determined based on the nature of the eigenvalues of the variation matrix.

Theorem 1: The trivial equilibrium point E_0 (0, 0, 0) is unstable.

Proof: The eigenvalues of the variation matrix (J_0) at E_0 are given by;

$det((J_0 - \lambda I) = 0$. Hence, $\lambda_1 = a_1, \lambda_2 = a_2, \lambda_3 = a_3$. Since $a_1, a_2, a_3 > 0$. Thus, the washed out state E_0 is unstable.

Theorem 2: States in which two species are washed out are unstable, i.e, E_1, E_2 and E_3.

Proof: The characteristic equations at E_1 is;

$$\left[a_1 + \frac{aa_3}{a_{33}} - \lambda\right]\left[a_2 - \frac{a_{23}a_3}{a_{33}} - \lambda\right][-a_3 - \lambda] = 0$$

The eigenvalues of this equation becomes; $\lambda_1 = a_1 + \frac{aa_3}{a_{33}}, \lambda_2 = a_2 - \frac{a_{23}a_3}{a_{33}}, \lambda_3 = -a_3$.

Since $\lambda_1 > 0$ unconditional positive. Hence, the steady state E_1 is unstable.

The characteristic equations at E_2 is solved as;

$$\left[a - \frac{a_{12}a_2}{a_{22}} - \lambda\right][-a_2 - \lambda][a_3 - \lambda] = 0$$

The eigenvalues of this equation becomes; $\lambda_1 = a - \frac{a_{12}a_2}{a_{22}}, \lambda_2 = -a_2, \lambda_3 = a_3$.

Since $\lambda_3 > 0$ unconditional positive. Hence, the steady state E_2 is unstable.

The characteristic equations at E_2 is solved as;

$$[-a_1 - \lambda]\left[a_2 + \frac{a_{21}a_1}{a_{11}} - \lambda\right][a_3 - \lambda] = 0$$

The eigenvalues of this equation becomes; $\lambda_1 = -a_1, \lambda_2 = a_2 + \frac{a_{21}a_1}{a_{11}}, \lambda_3 = a_3$.

As $\lambda_3 > 0$ unconditional positive. Hence, the equilibrium point E_3 is unstable.

Therefore, the states in which the two species washed out are unstable.

Theorem 3: States in which one species is washed out.

i. The steady states E_4, E_5 are stable under the following conditions.

That is E_4 is stable when $m < n$, where $m = a_1 + a_{22}a_{33} + aa_3a_{22} + a_{12}a_3a_{23}$,

$n = a_{12}a_2a_{33}$ and $a_3a_{23} < a_2a_{33}$ a nd E_5 is stable when $a_1 < 2a_{11}N_1^* + \left(\frac{aa_3}{a_{33}}\right)\left[\frac{1}{(1+ahN_1^*)^2}\right]$ and $a_2 + a_{21}N_1^* < \frac{a_3a_{23}}{a_{33}}$, where $N_1^* = \frac{\left(\frac{1}{ah} - \frac{a_1}{a_{22}a_{33}}\right) + \sqrt{\left(\frac{a_{22}a_{33} - aha_1}{aha_{22}a_{33}}\right)^2 - 4\left(\frac{a_1 + aa_3}{aha_{22}a_{33}}\right)}}{2}$

Proof: The Jacobian matrix (2.10) is evaluated at E_4 with the following eigenvalues:

$$J = \begin{pmatrix} a_1 + \frac{aa_3}{a_{33}} - a_{12}N_2^* & 0 & 0 \\ a_{21}N_2^* & a_2 - 2a_{22}N_2^* - a_{23}N_3^* & -a_{23}N_2^* \\ 0 & 0 & a_3 - 2a_{33}N_3^* \end{pmatrix}$$

The eigenvalues of $J(E_4)$ are obtained by solving the characteristic equation;

$$\left[a_1 + \frac{aa_3}{a_{33}} - a_{12}N_2^* - \lambda\right][a_2 - 2a_{22}N_2^* - a_{23}N_3^* - \lambda][a_3 - 2a_{33}N_3^* - \lambda] = 0, \text{ where,}$$

$$N_2^* = \frac{a_2}{a_{22}} - \frac{a_3a_{23}}{a_{22}a_{33}}, N_3^* = \frac{a_3}{a_{33}}.$$

Thus the eigenvalues are $\lambda_1 = \frac{m-n}{a_{22}a_3}$, $\lambda_2 = \frac{a_3a_{23}}{a_{33}} - a_2, \lambda_3 = -a_3$

Here, $\lambda_3 < 0$, conditional negative but $\lambda_1 < 0$ when $m < n$ and $\lambda_2 < 0$ when $a_3a_{23} < a_2a_{33}$.

Therefore, E_4 is stable under the set criteria, otherwise unstable.

Similarly, we can show E_5 as follows;

The evaluation of Jacobian matrix (2.10) at E_5 gives

$$J(E_5) = \begin{pmatrix} a_1 - 2a_{11}N_1^* + \frac{aN_3^*}{(1+ahN_1^*)^2} & -a_{12}N_1^* & \frac{aN_1^*}{1+ahN_1^*} \\ 0 & a_2 + a_{21}N_1^* - a_{23}N_3^* & 0 \\ 0 & 0 & a_3 - 2a_{33}N_3^* \end{pmatrix}$$

The characteristics equation of $J(E_5)$ is given by the following equation:

$$\left[a_1 - 2a_{11}N_1^* + \frac{aN_3^*}{(1+ahN_1^*)^2} - \lambda\right]$$

$$[a_2 + a_{21}N_1^* - a_{23}N_3^* - \lambda][a_3 - 2a_{33}N_3^* - \lambda] = 0$$

This gives, $\lambda_1 = a_1 - 2a_{11}N_1^* + \frac{aN_3^*}{(1+ahN_1^*)^2}$, $\lambda_2 = a_2 + a_{21}N_1^* - a_{23}N_3^*, \lambda_3 = a_3 - 2a_{33}N_3^*$.

Where $N_1^* = \frac{\left(\frac{1}{ah} - \frac{a_1}{a_{22}a_{33}}\right) + \sqrt{\left(\frac{a_{22}a_{33} - aha_1}{aha_{22}a_{33}}\right)^2 - 4\left(\frac{a_1 + aa_3}{aha_{22}a_{33}}\right)}}{2}$; $N_3^* = \frac{a_3}{a_{33}}$.

Here, $\lambda_3 < 0$ conditional negative and $\lambda_2 < 0$ when $a_2 + a_{21}N_1^* < \frac{a_3a_{23}}{a_{33}}$ but $\lambda_1 < 0$ when the following conditions are carefully holds. That is, $a_1 < 2a_{11}N_1^* + \frac{aN_3^*}{(1+ahN_1^*)^2}$ since N_1^* is positive quantity.

Hence, E_5 is locally stable under the set conditions, otherwise unstable.

ii. The steady state E_6 is unstable.

Proof: The evaluation of Jacobian matrix (2.10) at E_6 gives

$$J(E_6) = \begin{pmatrix} a_1 - 2a_{11}N_1^{**} - a_{12}N_2^{**} & -a_{12}N_1^* & \frac{aN_1^{**}}{1+ahN_1^{**}} \\ a_{21}N_2^{**} & a_2 - 2a_{22}N_2^{**} + a_{21}N_1^{**} & -a_{23}N_2^{**} \\ 0 & 0 & a_3 \end{pmatrix}$$

The characteristics equation of $J(E_6)$ is given by the following equation:

$$[a_1 - 2a_{11}N_1^{**} - a_{12}N_2^{**} - \lambda][a_2 - 2a_{22}N_2^{**} + a_{21}N_1^{**} - \lambda]$$

$$[a_3 - \lambda] + [a_{12}N_1^*][a_{21}N_2^{**}][a_3 - \lambda] = 0$$

This gives,

$$[a_1 - 2a_{11}N_1^{**} - a_{12}N_2^{**} - \lambda][a_2 - 2a_{22}N_2^{**} + a_{21}N_1^{**} - \lambda]$$

$$+[a_{12}N_1^*][a_{21}N_2^{**}] = 0, [a_3 - \lambda] = 0$$

Here, $\lambda_3 > 0$ *un* conditional positive. Thus E_6 is unstable whatever the case is.

Theorem 4: States in which the three species exists: $E_7(N_1^{***}, N_2^{***}, N_3^{***})$, is where

$$N_1^{***} = \frac{-P - \sqrt{P^2 - 4QS}}{2Q}; \ Q = -\left[a_{11}ah + \frac{a_{21}a_{12}ah}{a_{22}}\right],$$

$$P = a_1ah + \frac{a_3a_{12}a_{23}ah}{a_{22}a_{33}} - \left[\frac{a_{22}a_{11} + a_{12}a_2ah + a_{21}a_{12}}{a_{22}}\right],$$

$$S = a_1 + \frac{a_3a}{a_{33}} - \frac{a_{12}a_2}{a_{22}} + \frac{a_3a_{23}a_{12}}{a_{22}a_{33}}.$$

$$N_2^{***} = \frac{a_2}{a_{22}} + \frac{a_{21}}{a_{22}}N_1^{***} - \frac{a_3a_{23}}{a_{22}a_{33}};$$

$$N_3^{***} = \frac{a_3}{a_{33}}.$$

The Jacobian matrix at an equilibrium point $E_7 = (N_1^{***}, N_2^{***}, N_3^{***})$ is given by;

$$J = \begin{pmatrix} a_1 - 2a_{11}N_1^{***} + \frac{aN_3^{***}}{(1+ahN_1^{***})^2} - a_{12}N_2^{***} & -a_{12}N_1^{***} & \frac{aN_1^{***}}{1+ahN_1^{***}} \\ a_{21}N_2^{***} & a_2 - 2a_{22}N_2^{***} + a_{21}N_1^{***} - a_{23}N_3^{***} & -a_{23}N_2^{***} \\ 0 & 0 & a_3 - 2a_{33}N_3^{***} \end{pmatrix}$$

The characteristics equation of $J(E_7)$ is given by the following quadratic equation

$$\left[a_1 - 2a_{11}N_1^{***} + \frac{aN_3^{***}}{(1+ahN_1^{***})^2} - a_{12}N_2^{***} - \lambda\right]$$

$$[a_2 - 2a_{22}N_2^{***} + a_{21}N_1^{***} - a_{23}N_3^{***} - \lambda][a_3 - 2a_{33}N_3^{***} - \lambda] + (a_{12}N_1^{***})(a_{21}N_2^{***})(a_3 - 2a_{33}N_3^{***} - \lambda) = 0$$

This implies,

$$\left[a_1 - 2a_{11}N_1^{***} + \frac{aN_3^{***}}{(1+ahN_1^{***})^2} - a_{12}N_2^{***} - \lambda\right][a_2 - 2a_{22}N_2^{***} + a_{21}N_1^{***} - a_{23}N_3^{***} - \lambda][a_3 - 2a_{33}N_3^{***} - \lambda] + (a_{12}N_1^{***})(a_{21}N_2^{***}) = 0; (a_3 - 2a_{33}N_3^{***} - \lambda) = 0$$

$$H\lambda^2 + (A+B)\lambda + D = 0$$

Where $A = 2a_{11}N_1^{***} - \frac{aN_3^{***}}{(1+ahN_1^{***})^2} + a_{12}N_2^{***} - a_1, B = 2a_{22}N_2^{***} - a_{21}N_1^{***} + a_{23}N_3^{***} - a_2,$

$$D = AB + (a_{12}N_1^{***})(a_{21}N_2^{***})$$

From the Routh-Hurwitz criterion, we can conclude that $H = 1 > 0, (A+B) > 0$ *or* $D > 0$

Hence, E_7 is locally asymptotically stable.

5.2. Global Stability of Steady State

The main goal of this model formulation is targeted to eradicate the enemy. Mathematically, this can be achieved whenever the second species free equilibrium is stable. The sufficient condition for this equilibrium to be globally asymptotically stable is given by the following theorem.

Theorem 5: If $a_2 + a_{21}N_1^* < \frac{a_3 a_{23}}{a_{33}}$ and $a_1 < 2a_{11}N_1^* + \left(\frac{aa_3}{a_{33}}\right)\left(\frac{1}{(1+ahN_1^*)^2}\right)$ the steady states E_5 is globally asymptotically stable.

Proof: Consider the Lyapunov function derived from the integral form;

$$\int_{x^*}^{x} \frac{u - x^*}{u} du$$

Now, let $V(N_1, N_2, N_3) = N_1 - N_1^* - N_1^* \ln\frac{N_1}{N_1^*} + l\left(N_3 - N_3^* - N_3^* \ln\frac{N_3}{N_3^*}\right)$, where l is some positive constant assumed.

Now, the differential of v with respect to t and after some algebraic manipulations reduces to the following form:

$$\dot{V}(N_1, N_2, N_3) = \left[\frac{N_1 - N_1^*}{N_1}\right]\left(\frac{dN_1}{dt}\right) + l\left[\frac{N_3 - N_3^*}{N_3}\right]\left(\frac{dN_3}{dt}\right)$$

$$= \left[\frac{N_1 - N_1^*}{N_1}\right]\left[a_1 N_1 - a_{11}N^2{}_1 + \frac{a}{1 + ahN_1}N_1 N_3 - a_{12}N_1 N_2\right] + l\left[\frac{N_3 - N_3^*}{N_3}\right][a_3 N_3 - a_{33}N^2{}_3]$$

$$= [N_1 - N_1^*]\left[a_1 - a_{11}N_1 + \frac{aN_3}{1 + ahN_1}\right] + l[N_3 - N_3^*][a_3 - a_{33}N_3]$$

$$= -[N_1 - N_1^*]^2\left(\frac{a_1}{a_{11}} - \omega\right) + -l[N_3 - N_3^*]^2\left[\frac{a_3}{a_{33}}\right]$$

$$= -\left[(N_1 - N_1^*)^2\left(\frac{a_1}{a_{11}} - \omega\right) + l[N_3 - N_3^*]^2\left(\frac{a_3}{a_{33}}\right)\right]$$

Choosing, $l = \frac{a_{33}}{a_3}$ and letting $\omega = \frac{aN_3}{1+ahN_1}$ for the sake of simplicity.

Thus, $\frac{dV}{dt} < 0$, i.e., v is positive definite and also $\dot{V}(N_1^*, N_2^*, N_3^*) = 0$. Therefore E_5 is globally asymptotically stable.

6. Numerical Simulations and Discussion.

To illustrate the dynamical behavior of system (2.1), (2.2), and (2.3), we perform some numerical simulations using hypothetical value of parameters.

The parameters and its values used in this study are mentioned in the following table as follow

Table 3. *Parameters and their estimated values for figures (2-6).*

Figures	Estimated Parameters and their values											Remark
	a_1	a_{11}	a	h	a_{12}	a_2	a_{22}	a_{21}	a_{23}	a_3	a_{33}	
6.1	0.0000	0.0390	0.0000	0.0000	0.9740	0.0000	0.1430	0.0100	0.6010	0.0260	0.9740	[16]
6.2	0.1000	0.0710	0.3000	0.9000	0.0100	0.6620	0.1170	0.0900	0.4640	0.9350	0.0650	Assumed
6.3	0.0420	0.2790	0.1200	0.5100	0.8770	0.7110	0.5580	0.9450	0.0840	0.4610	0.2180	Assumed
6.4	0.9350	0.2790	0.1200	0.5100	0.8770	0.9220	0.9320	0.9450	0.9220	0.9680	0.9680	Assumed
6.5	0.3250	0.0260	0.3150	0.1690	0.4970	0.1200	0.0780	0.9610	0.9610	0.9450	0.1790	Assumed

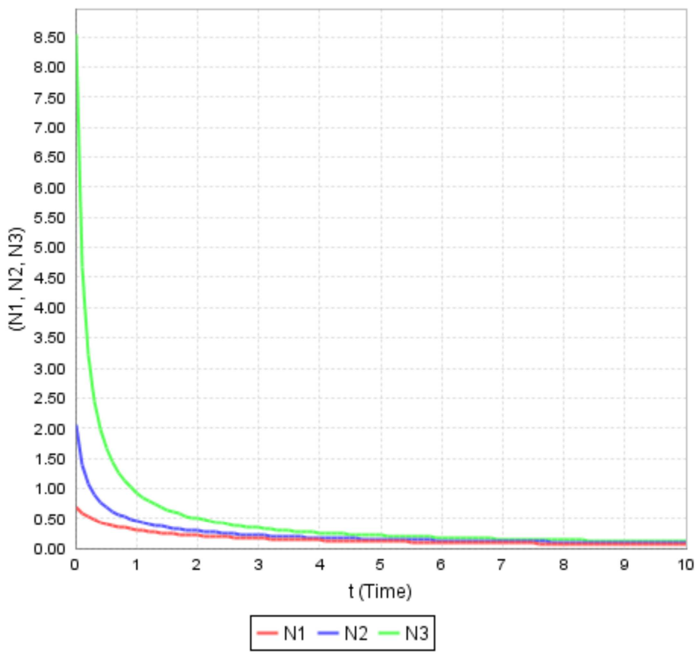

Figure 2. *The dynamic behavior of the system with different initial condition and the three species washed out.*

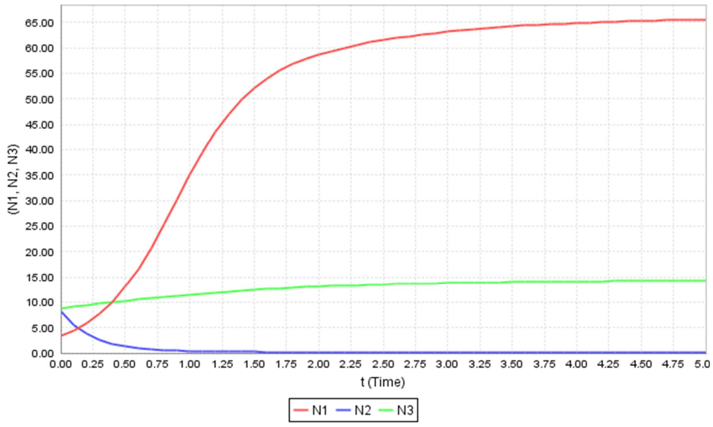

Figure 3. *The dynamic behavior of the system with different initial condition and the second species washed out while the first and the third species exist.*

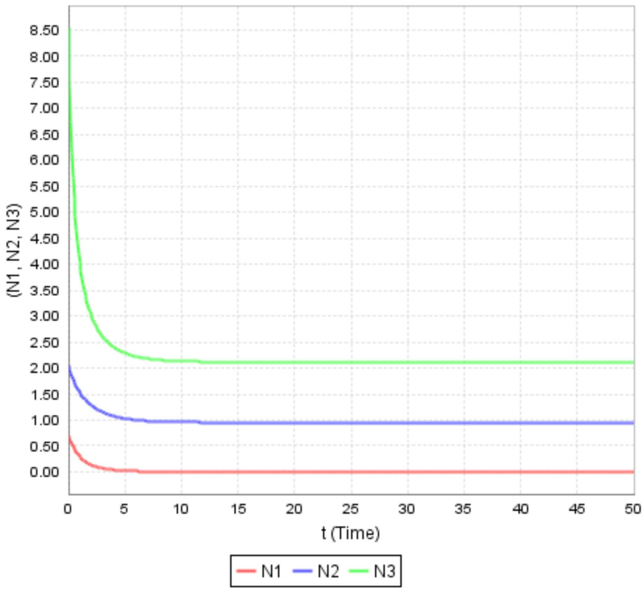

Figure 4. *The dynamic behavior of the system with different initial condition and the second and the third species exists while the first species washed out.*

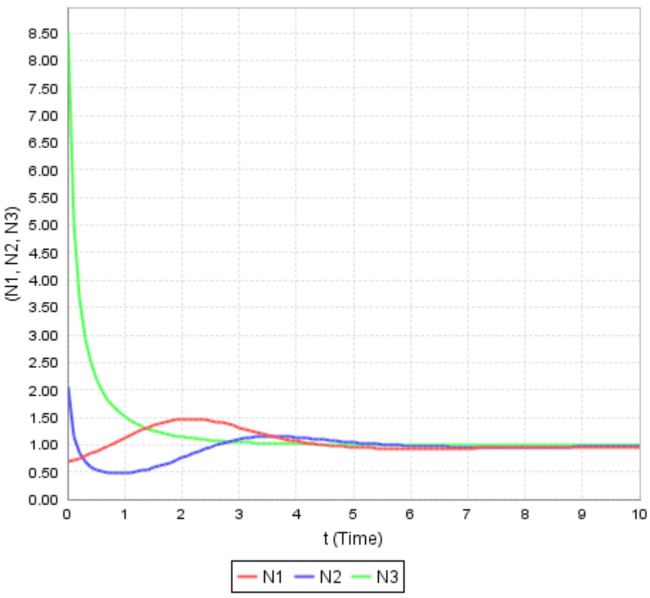

Figure 5. *The dynamic behavior of the system with different initial condition and the three species exists.*

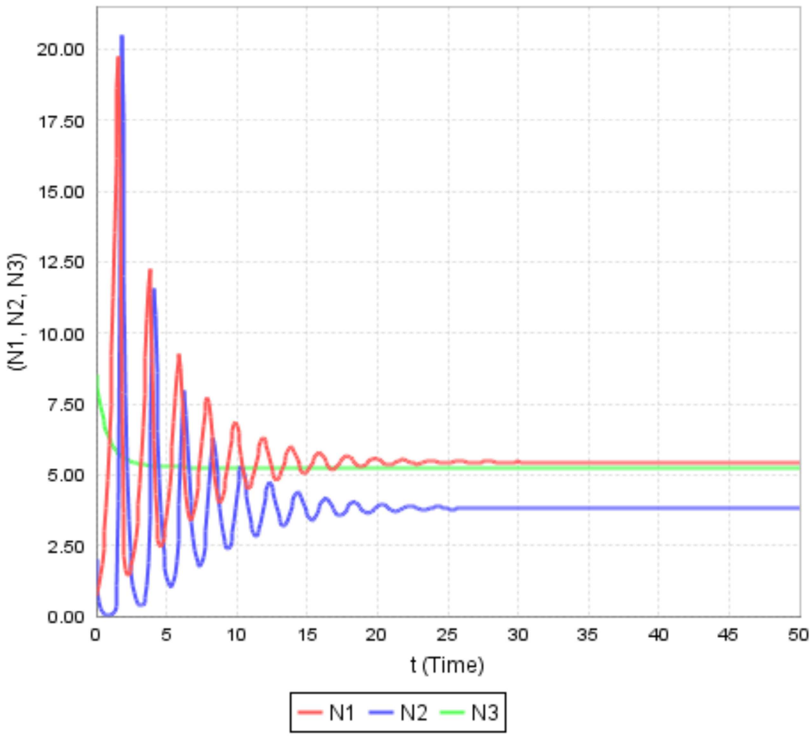

Figure 6. *The dynamic behavior of the system with initial condition varying and certain disturbance between the species and living together after certain time t.*

7. Conclusion and Future Extension

In this paper a three species ecosystem with various interactions between the species is considered for investigation. The biological feasible of the system was shown using different mathematical tools, like positivity and boundedness. The equilibrium points are examined. The stability of some the steady states were investigated.

It is observed that among the states, the state in which the first species and the second species washed out (extinct), is asymptotically stable as shown in figure 3 and figure 4. From this it can be understood that in the former case, though the first and second species are extinct, the third species survives due to non inhibition by its enemy and the response taken between the species. While in the later case, the species continue to exist together because the response taken between the first species and the third species with Monod functional response force them for its survival. The results are illustrated in figures 5 and 6.

In another word we try to incorporate the Monod type response term for the first species of the system given in the model, this seems more interesting and necessary, since more and more species become endangered due to the over exploitation by humans or environmental change. It is shown in figures 3 - 6 that the commensalism of the second species to the first species could avoid the extinction of the species. Moreover, if the cooperative intensity is large enough, then the two species could really coexist in a stable state in which the third species could avoid the decline of the other species. However, if the effect of the response of the third species is limited, it has been shown that, the first species still be driven to extinction. In addition to this there is a dispensation or allee effect of the third species, because of over exploitation.

These models can be further extended by other type of response functions and selective harvesting by combined harvesting and also constant rate of harvesting by variable rate of harvesting. This is our further investigation.

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